

# ANALYSIS OF MIXED INTERIOR PENALTY DISCONTINUOUS GALERKIN METHODS FOR THE CAHN-HILLIARD EQUATION AND THE HELE-SHAW FLOW

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**Abstract.** This paper proposes and analyzes two fully discrete mixed interior penalty discontinuous Galerkin (DG) methods for the fourth order nonlinear Cahn-Hilliard equation. Both methods use the backward Euler method for time discretization and interior penalty discontinuous Galerkin methods for spatial discretization. They differ from each other on how the nonlinear term is treated, one of them is based on fully implicit time-stepping and the other uses the energy-splitting time-stepping. The primary goal of the paper is to prove the convergence of the numerical interfaces of the DG methods to the interface of the Hele-Shaw flow. This is achieved by establishing error estimates that depend on  $\epsilon^{-1}$  only in some low polynomial orders, instead of exponential orders. Similar to [14], the crux is to prove a discrete spectrum estimate in the discontinuous Galerkin finite element space. However, the validity of such a result is not obvious because the DG space is not a subspace of the (energy) space  $H^1(\Omega)$  and it is larger than the finite element space. This difficult is overcome by a delicate perturbation argument which relies on the discrete spectrum estimate in the finite element space proved in [14]. Numerical experiment results are also presented to gauge the theoretical results and the performance of the proposed fully discrete mixed DG methods.

**Key words.** Cahn-Hilliard equation, Hele-Shaw problem, phase transition, discontinuous Galerkin method, discrete spectral estimate, convergence of numerical interface.

**AMS subject classifications.** 65N12, 65N15, 65N30,

**1. Introduction.** This paper concerns with mixed interior penalty discontinuous Galerkin (MIP-DG) approximations of the following Cahn-Hilliard problem:

$$(1.1) \quad u_t - \Delta w = 0 \quad \text{in } \Omega_T := \Omega \times (0, T),$$

$$(1.2) \quad -\epsilon \Delta u + \frac{1}{\epsilon} f(u) = w \quad \text{in } \Omega_T,$$

$$(1.3) \quad \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega_T := \partial\Omega \times (0, T),$$

$$(1.4) \quad u = u_0 \quad \text{in } \Omega \times \{t = 0\}.$$

Here  $\Omega \subseteq \mathbf{R}^d$  ( $d = 2, 3$ ) is a bounded domain, and  $f(u) = F'(u)$ ,  $F(u)$  is a nonconvex potential density function which takes its global minimum zero at  $u = \pm 1$ . In this paper, we only consider the following quartic potential density function:

$$(1.5) \quad F(u) = \frac{1}{4}(u^2 - 1)^2.$$

After eliminating the intermediate variable  $w$  (called the chemical potential), the above system reduces into a fourth order nonlinear PDE for  $u$ , which is known as the

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Cahn-Hilliard equation in the literature. This equation was originally introduced by John W. Cahn and John E. Hilliard in [5] to describe the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. Here  $u$  and  $1 - u$  denote respectively the concentrations of the two fluids, with  $u = \pm 1$  indicating domains of the two components. We note that the equation (1.1)–(1.2) differs from the original Cahn-Hilliard equation in the scaling of the time, and  $t$  here corresponds to  $\frac{t}{\epsilon}$  in the original formulation.  $\epsilon$ , which is positively small, is called the interaction length.

Besides its important role in materials phase transition, the Cahn-Hilliard equation has been extensively studied due to its close relation with the Hele-Shaw problem. It was first formally proved by Pego [19] that the chemical potential  $w := -\epsilon \Delta u + \frac{1}{\epsilon} f(u)$  tends to a limit which satisfies the following free boundary problem known as the Hele-Shaw problem:

$$(1.6) \quad \Delta w = 0 \quad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0, T],$$

$$(1.7) \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad t \in [0, T],$$

$$(1.8) \quad w = \sigma \kappa \quad \text{on } \Gamma_t, \quad t \in [0, T],$$

$$(1.9) \quad V = \frac{1}{2} \left[ \frac{\partial w}{\partial n} \right]_{\Gamma_t} \quad \text{on } \Gamma_t, \quad t \in [0, T],$$

as  $\epsilon \searrow 0$ , provided that the Hele-Shaw problem has a unique classical solution. Here

$$(1.10) \quad \sigma = \int_{-1}^1 \sqrt{\frac{F(s)}{2}} ds.$$

$\kappa$  and  $V$  represent the mean curvature and the normal velocity of the interface  $\Gamma_t$ . A rigorous justification that  $u \rightarrow \pm 1$  in the interior or exterior of  $\Gamma_t$  for all  $t \in [0, T]$  as  $\epsilon \searrow 0$  was given by Stoth [22] for the radially symmetric case, and by Alikakos, Bates and Chen [2] for the general case. In addition, Chen [7] established the convergence of the weak solution of the Cahn-Hilliard problem to a weak (or varifold) solution of the Hele-Shaw problem.

Moreover, the Cahn-Hilliard equation (together with the Allen-Cahn equation) has become a fundamental equation as well as a building block in the phase field methodology (or the diffuse interface methodology) for moving interface and free boundary problems arising from various applications such as fluid dynamics, materials science, image processing and biology (cf. [20, 12] and the references therein). The diffuse interface approach provides a convenient mathematical formalism for numerically approximating the moving interface problems because explicitly tracking the interface is not needed in the diffuse interface formulation. The main advantage of the diffuse interface method is its ability to handle with ease singularities of the interfaces. Like many singular perturbation problems, the main computational issue is to resolve the (small) scale introduced by the parameter  $\epsilon$  in the equation. Computationally, the problem could become intractable, especially in three-dimensional cases if uniform meshes are used. This difficulty is often overcome by exploiting the predictable (at least for small  $\epsilon$ ) PDE solution profile and by using adaptive mesh techniques (cf. [16] and the references therein), so fine meshes are only used in the diffuse interface region.

Numerical approximations of the Cahn-Hilliard equation have been extensively carried out in the past thirty years (cf. [9, 11, 14] and the references therein). On

the other hand, the majority of these works were done for a fixed parameter  $\epsilon$ . The error bounds, which are obtained using the standard Gronwall inequality technique, show an exponential dependence on  $1/\epsilon$ . Such an estimate is clearly not useful for small  $\epsilon$ , in particular, in addressing the issue whether the computed numerical interfaces converge to the original sharp interface of the Hele-Shaw problem. Better and practical error bounds should only depend on  $1/\epsilon$  in some (low) polynomial orders because they can be used to provide an answer to the above convergence question, which in fact is the best result (in terms of  $\epsilon$ ) one can expect. The first such polynomial order in  $1/\epsilon$  a priori estimate was obtained in [15] for mixed finite element approximations of the Cahn-Hilliard problem (1.1)–(1.5). In addition, polynomial order in  $1/\epsilon$  a posteriori error estimates were obtained in [16] for the same mixed finite element methods. One of the key ideas employed in all these works is to use a non-standard error estimate technique which is based on establishing a discrete spectrum estimate (using its continuous counterpart) for the linearized Cahn-Hilliard operator. An immediate corollary of the polynomial order in  $1/\epsilon$  a priori and a posteriori error estimates is the convergence of the numerical interfaces of the underlying mixed finite element approximations to the Hele-Shaw flow before the onset of singularities of the Hele-Shaw flow as  $\epsilon$  and mesh sizes  $h$  and  $k$  all tend to zero.

The objectives of this paper are twofold: Firstly, we develop some MIP-DG methods and to establish polynomial order in  $1/\epsilon$  a priori error bounds, as well as to prove convergence of numerical interfaces for the MIP-DG methods. This goal is motivated by the advantages of DG methods in regard to designing adaptive mesh methods and algorithms, which is an indispensable strategy with the diffuse interface methodology. Secondly, we use the Cahn-Hilliard equation as another prototypical model problem [13] to develop new analysis techniques for analyzing convergence of numerical interfaces to the underlying sharp interface for DG (and nonconforming finite element) discretizations of phase field models. To the best of our knowledge, no such convergence result and analysis technique is available in the literature for fourth order PDEs. The main obstacle for improving the finite element techniques of [15] is that the DG (and nonconforming finite element) spaces are not subspaces of  $H^1(\Omega)$ . As a result, whether the needed discrete spectrum estimate holds becomes a key question to answer.

This paper consists of four additional sections. In section 2 we first collect some a priori error estimates for problem (1.1)–(1.5), which show the explicit dependence on the parameter  $\epsilon$ . We then cite two important technical lemmas to be used in the later sections. One of the lemma states the spectral estimate for the linearized Cahn-Hilliard operator. In section 3, we propose two fully discrete MIP-DG schemes for problem (1.1)–(1.5), they differ only in their treatment of the nonlinear term. The first main result of this section is to establish a discrete spectrum estimate in the DG space, which mimics the spectral estimates for the differential operator and its finite element counterpart. The second main result of this section is to derive optimal error bounds which depends on  $1/\epsilon$  only in low polynomial orders for both fully discrete MIP-DG methods. In section 4, using the refined error estimates of section 3, we prove the convergence of the numerical interfaces of the fully discrete MIP-DG methods to the interface of the Hele-Shaw flow before the onset of the singularities as  $\epsilon, h$  and  $k$  all tend to zero. Finally, in section 5 we provide some numerical experiments to gauge the performance of the proposed fully discrete MIP-DG methods.

**2. Preliminaries.** In this section, we shall collect some known results about problem (1.1)–(1.5) from [6, 14, 15], which will be used in sections 3 and 4. Some

general assumptions on the initial condition, as well as some energy estimates based on these assumptions, will be cited. Standard function and space notations are adopted in this paper [1, 4]. We use  $(\cdot, \cdot)$  and  $\|\cdot\|_{L^2}$  to denote the standard inner product and norm on  $L^2(\Omega)$ . Throughout this paper,  $C$  denotes a generic positive constant independent of  $\epsilon$ , space and time step sizes  $h$  and  $k$ , which may have different values at different occasions.

We begin with the following well known fact [2] that the Cahn-Hilliard equation (1.1)-(1.5) can be interpreted as the  $H^{-1}$ -gradient flow for the Cahn-Hilliard energy functional

$$(2.1) \quad J_\epsilon(v) := \int_{\Omega} \left( \frac{\epsilon}{2} |\nabla v|^2 + \frac{1}{\epsilon} F(v) \right) dx$$

The following assumptions on the initial datum  $u_0$  were made in [14], they were used to derive a priori estimates for the solution of problem (1.1)-(1.5).

**General Assumption (GA)**

- (1) Assume that  $m_0 \in (-1, 1)$  where

$$(2.2) \quad m_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx.$$

- (2) There exists a nonnegative constant  $\sigma_1$  such that

$$(2.3) \quad J_\epsilon(u_0) \leq C\epsilon^{-2\sigma_1}.$$

- (3) There exists nonnegative constants  $\sigma_2, \sigma_3$  and  $\sigma_4$  such that

$$(2.4) \quad \left\| -\epsilon \Delta u_0 + \epsilon^{-1} f(u_0) \right\|_{H^\ell(\Omega)} \leq C\epsilon^{-\sigma_2+\ell}, \quad \ell = 0, 1, 2.$$

Under the above assumptions, the following solution estimates were proved in [14, 15].

**PROPOSITION 2.1.** *The solution  $u$  of problem (1.1)-(1.5) satisfies the following energy estimates:*

$$(2.5) \quad \operatorname{ess\,sup}_{t \in [0, \infty)} \left( \frac{\epsilon}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{\epsilon} \|F(u)\|_{L^1} \right) + \left\{ \int_0^\infty \|u_t(s)\|_{H^{-1}}^2 ds + \int_0^\infty \|\nabla w(s)\|_{L^2}^2 ds \right\} \leq J_\epsilon(u_0),$$

$$(2.6) \quad \operatorname{ess\,sup}_{t \in [0, \infty)} \|u\|_{L^4}^4 \leq C(1 + J_\epsilon(u_0)),$$

$$(2.7) \quad \operatorname{ess\,sup}_{t \in [0, \infty)} \|u^2 - 1\|_{L^2}^2 \leq C\epsilon J_\epsilon(u_0).$$

Moreover, suppose that (2.2)-(2.4) hold,  $u_0 \in H^4(\Omega)$  and  $\partial\Omega \in C^{2,1}$ , then  $u$  satisfies

the additional estimates:

$$(2.8) \quad \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx = m_0 \quad \forall t \geq 0,$$

$$(2.9) \quad \int_0^\infty \|\Delta u\|_{L^2}^2 ds \leq C\epsilon^{-(2\sigma_1+3)},$$

$$(2.10) \quad \int_0^\infty \|\nabla \Delta u\|_{L^2}^2 ds \leq C\epsilon^{-(2\sigma_1+5)},$$

$$(2.11) \quad \begin{cases} \text{ess sup}_{t \in [0, \infty)} \|u_t\|_{H^{-1}}^2 \\ \text{ess sup}_{t \in [0, \infty)} \|\nabla w\|_{L^2}^2 \end{cases} + \epsilon \int_0^\infty \|\nabla u_t\|_{L^2}^2 ds \leq C\epsilon^{-\max\{2\sigma_1+3, 2\sigma_3\}},$$

$$(2.12) \quad \text{ess sup}_{t \in [0, \infty)} \|\Delta u\|_{L^2} \leq C\epsilon^{-\max\{\sigma_1+\frac{5}{2}, \sigma_3+1\}},$$

$$(2.13) \quad \text{ess sup}_{t \in [0, \infty)} \|\nabla \Delta u\|_{L^2} \leq C\epsilon^{-\max\{\sigma_1+\frac{5}{2}, \sigma_3+1\}},$$

$$(2.14) \quad \begin{cases} \int_0^\infty \|u_t\|_{L^2}^2 ds \\ \int_0^\infty \|\Delta w\|_{L^2}^2 ds \end{cases} + \text{ess sup}_{t \in [0, \infty)} \epsilon \|\Delta u\|_{L^2}^2 \leq C\epsilon^{-\max\{2\sigma_1+\frac{7}{2}, 2\sigma_3+\frac{1}{2}, 2\sigma_2+1\}},$$

$$(2.15) \quad \epsilon \int_0^\infty \|\Delta u_t\|_{L^2}^2 ds + \text{ess sup}_{t \in [0, \infty)} \|u_t\|_{L^2}^2 \leq C\epsilon^{-\max\{2\sigma_1+\frac{13}{2}, 2\sigma_3+\frac{7}{2}, 2\sigma_2+4, 2\sigma_4\}},$$

$$(2.16) \quad \int_0^\infty \|\Delta^{-1} u_{tt}(s)\|_{H^{-1}}^2 ds \leq C\epsilon^{-\max\{10\sigma_1+10, 4\sigma_1+2\sigma_2+5, 2\sigma_3-1\}}.$$

Furthermore, if there exists  $\sigma_5 > 0$  such that

$$(2.17) \quad \lim_{s \rightarrow 0^+} \|\nabla u_t(s)\|_{L^2} \leq C\epsilon^{-\sigma_5},$$

then there hold for  $d = 2, 3$ ,

$$(2.18) \quad \text{ess sup}_{t \in [0, \infty)} \|\nabla u_t\|_{L^2}^2 + \epsilon \int_0^\infty \|\nabla \Delta u_t\|_{L^2}^2 ds \leq C\rho_0(\epsilon, d),$$

$$(2.19) \quad \int_0^\infty \|u_{tt}\|_{H^{-1}}^2 ds \leq C\rho_1(\epsilon, d),$$

$$(2.20) \quad \text{ess sup}_{t \in [0, \infty)} \|\Delta^2 u\|_{L^2} \leq C\rho_2(\epsilon),$$

where

$$\begin{aligned} \rho_0(\epsilon, d) &:= \epsilon^{-\frac{2}{6-d} \max\{2\sigma_1+5, 2\sigma_3+2\} - \max\{2\sigma_1+\frac{13}{2}, 2\sigma_3+\frac{7}{2}, 2\sigma_2+4\}} + \epsilon^{-2\sigma_5} \\ &\quad + \epsilon^{-\max\{2\sigma_1+7, 2\sigma_3+4\}}, \\ \rho_1(\epsilon, d) &:= \epsilon\rho_0(\epsilon, d), \\ \rho_2(\epsilon) &:= \epsilon^{-\max\{\sigma_1+5, \sigma_3+\frac{7}{2}, \sigma_2+\frac{5}{2}, \sigma_4+1\}}. \end{aligned}$$

The next lemma concerns with a lower bound estimate for the principal eigenvalue of the linearized Cahn-Hilliard operator, a proof of this lemma can be found in [6].

LEMMA 2.2. *Suppose that (2.2)–(2.4) hold. Given a smooth initial curve/surface  $\Gamma_0$ , let  $u_0$  be a smooth function satisfying  $\Gamma_0 = \{x \in \Omega; u_0(x) = 0\}$  and some profile*

described in [6]. Let  $u$  be the solution to problem (1.1)–(1.5). Define  $\mathcal{L}_{CH}$  as

$$(2.21) \quad \mathcal{L}_{CH} := \Delta \left( \epsilon \Delta - \frac{1}{\epsilon} f'(u) I \right).$$

Then there exists  $0 < \epsilon_0 \ll 1$  and a positive constant  $C_0$  such that the principle eigenvalue of the linearized Cahn-Hilliard operator  $\mathcal{L}_{CH}$  satisfies

$$(2.22) \quad \lambda_{CH} := \inf_{\substack{0 \neq \psi \in H^1(\Omega) \\ \Delta w = \psi}} \frac{\epsilon \|\nabla \psi\|_{L^2}^2 + \frac{1}{\epsilon} (f'(u) \psi, \psi)}{\|\nabla w\|_{L^2}^2} \geq -C_0$$

for  $t \in [0, T]$  and  $\epsilon \in (0, \epsilon_0)$ .

REMARK 1. (a) A discrete generalization of (2.22) on  $C^0$  finite element spaces was proved in [14, 15]. It plays a pivotal role in the nonstandard convergence analysis of [14, 15]. In the next section, we shall prove another discrete generalization of (2.12) on the DG finite element space.

(b) The restriction on the initial function  $u_0$  is needed to guarantee that the solution  $u(t)$  satisfies certain profile at later time  $t > 0$  which is required in the proof of [6]. One example of admissible initial functions is  $u_0 = \tanh(\frac{d_0(x)}{\epsilon})$ , where  $d_0(x)$  stands for the signed distance function to the initial interface  $\Gamma_0$ . Such a  $u_0$  is smooth when  $\Gamma_0$  is smooth.

Next lemma can be regarded as a nonlinear generalization of the classical discrete Gronwall lemma. It gives an upper bound estimate for a discrete sequence which satisfies a nonlinear inequality with Bernoulli-type nonlinearity, which will be utilized crucially in the next section. A proof of this lemma can be found in [18] and its differential counterpart can be seen in [16].

LEMMA 2.3. Let  $\{S_\ell\}_{\ell \geq 1}$  be a positive nondecreasing sequence,  $\{b_\ell\}_{\ell \geq 1}$  and  $\{k_\ell\}_{\ell \geq 1}$  be nonnegative sequences, and  $p > 1$  be a constant. If

$$(2.23) \quad S_{\ell+1} - S_\ell \leq b_\ell S_\ell + k_\ell S_\ell^p \quad \text{for } \ell \geq 1,$$

$$(2.24) \quad S_1^{1-p} + (1-p) \sum_{s=1}^{\ell-1} k_s a_{s+1}^{1-p} > 0 \quad \text{for } \ell \geq 2,$$

then

$$(2.25) \quad S_\ell \leq \frac{1}{a_\ell} \left\{ S_1^{1-p} + (1-p) \sum_{s=1}^{\ell-1} k_s a_{s+1}^{1-p} \right\}^{\frac{1}{1-p}} \quad \text{for } \ell \geq 2,$$

where

$$(2.26) \quad a_\ell := \prod_{s=1}^{\ell-1} \frac{1}{1 + b_s}.$$

**3. Fully discrete MIP-DG approximations.** In this section we present and analyze two fully discrete MIP-DG methods for the Cahn-Hilliard problem (1.1)–(1.5). The primary goal of this section is to derive error estimates for the DG solutions that depend on  $\epsilon^{-1}$  only in low polynomial orders, instead of exponential orders. As in the finite element case (cf. [15]), the crux is to establish a discrete spectrum estimate for the linearized Cahn-Hilliard operator on the DG space.

**3.1. Formulations of the MIP-DG method.** Let  $\mathcal{T}_h = \{K\}_{K \in \Omega}$  be a quasi-uniform triangulation of  $\Omega$  parameterized by  $h > 0$ . For any triangle/tetrahedron  $K \in \mathcal{T}_h$ , we define  $h_K$  to be the diameter of  $K$ , and  $h := \max_{K \in \mathcal{T}_h} h_K$ . The standard broken Sobolev space is defined as

$$(3.1) \quad H^s(\mathcal{T}_h) := \{v \in L^2(\Omega); \forall K \in \mathcal{T}_h, v|_K \in H^s(K)\}.$$

For any  $K \in \mathcal{T}_h$ ,  $P_r(K)$  denotes the set of all polynomials of degree at most  $r(\geq 1)$  on the element  $K$ , and the DG finite element space  $V_h$  is defined as

$$(3.2) \quad V_h := \{v \in L^2(\Omega); \forall K \in \mathcal{T}_h, v|_K \in P_r(K)\}.$$

Let  $L_0^2$  denote the set of functions in  $L^2(\Omega)$  with zero mean, and let  $\mathring{V}_h := V_h \cap L_0^2$ . We also define  $\mathcal{E}_h^I$  to be the set of all interior edges/faces of  $\mathcal{T}_h$ ,  $\mathcal{E}_h^B$  to be the set of all boundary edges/faces of  $\mathcal{T}_h$  on  $\Gamma = \partial\Omega$ , and  $\mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^B$ . Let  $e$  be an interior edge shared by two elements  $K_1$  and  $K_2$ . For a scalar function  $v$ , define

$$\{v\} = \frac{1}{2}(v|_K + v|_{K'}), \quad [v] = v|_K - v|_{K'}, \quad \text{on } e \in \mathcal{E}_h^I,$$

where  $K$  is  $K_1$  or  $K_2$ , whichever has the bigger global labeling and  $K'$  is the other. The  $L^2$ -inner product for piecewise functions over the mesh  $\mathcal{T}_h$  is naturally defined by

$$(v, w)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \int_K v w dx.$$

Let  $0 \leq t_0 < t_1 < \dots < t_M = T$  be a partition of the interval  $[0, T]$  with time step  $k = t_{n+1} - t_n$ . Our fully discrete MIP-DG methods are defined as follows: for any  $1 \leq m \leq M$ ,  $(U^m, W^m) \in V_h \times V_h$  are given by

$$(3.3) \quad (d_t U^m, \eta) + a_h(W^m, \eta) = 0 \quad \forall \eta \in V_h,$$

$$(3.4) \quad \epsilon a_h(U^m, v) + \frac{1}{\epsilon}(f^m, v) - (W^m, v) = 0 \quad \forall v \in V_h,$$

where

$$(3.5) \quad \begin{aligned} a_h(u, v) = & \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v dx - \sum_{e \in \mathcal{E}_h^I} \int_e \{\nabla u \cdot \mathbf{n}_e\} [v] ds \\ & - \sum_{e \in \mathcal{E}_h^I} \int_e \{\nabla v \cdot \mathbf{n}_e\} [u] ds + \sum_{e \in \mathcal{E}_h^I} \int_e \frac{\sigma_e^0}{h_e} [u] [v] ds, \end{aligned}$$

and  $\sigma_e^0 > 0$  is the penalty parameter. There are two choices of  $f^m$  considered in this paper, namely

$$f^m = (U^m)^3 - U^{m-1} \quad \text{and} \quad f^m = (U^m)^3 - U^m,$$

which lead to the energy-splitting scheme and fully implicit scheme respectively.  $d_t$  is the (backward) difference operator defined by  $d_t U^m := (U^m - U^{m-1})/k$  and  $U^0 := \hat{P}_h u_0$  (or  $\hat{Q}_h u_0$ ) is the starting value, with the finite element  $H^1$  (or  $L^2$ ) projection  $\hat{P}_h$  (or  $\hat{Q}_h$ ) to be defined below. We refer to [13] for a discussion why a continuous projection is needed for the initial condition. We remark that only the fully implicit case was considered in [14, 15] for the mixed finite element method.

In order to analyze the stability of (3.3)–(3.4), we need some preparations. First, we introduce three projection operators that will be needed to derive the error estimates in section 3.4.  $P_h : H^s(\mathcal{T}_h) \rightarrow V_h$  denotes the elliptic projection operator defined by

$$(3.6) \quad a_h(u - P_h u, v_h) + (u - P_h u, v_h) = 0 \quad \forall v_h \in V_h,$$

which has the following approximation properties (see [8]):

$$(3.7) \quad \|v - P_h v\|_{L^2(\mathcal{T}_h)} + h\|\nabla(v - P_h v)\|_{L^2(\mathcal{T}_h)} \leq Ch^{\min\{r+1, s\}} \|u\|_{H^s(\mathcal{T}_h)},$$

$$(3.8) \quad \frac{1}{|\ln h|^{\bar{r}}} \|v - P_h v\|_{L^\infty(\mathcal{T}_h)} + h\|\nabla(u - P_h u)\|_{L^\infty(\mathcal{T}_h)} \leq Ch^{\min\{r+1, s\}} \|u\|_{W^{s, \infty}(\mathcal{T}_h)}.$$

Here  $\bar{r} := \min\{1, r\} - \min\{1, r - 1\}$ .

Let  $\hat{P}_h : H^s(\mathcal{T}_h) \rightarrow S_h := V_h \cap C^0(\bar{\Omega})$  denote the standard continuous finite element elliptic projection, which is the counterpart of projection  $P_h$ . It has the following well-known property [14, 15]:

$$(3.9) \quad \|u - \hat{P}_h u\|_{L^\infty} \leq Ch^{2-\frac{d}{2}} \|u\|_{H^2}.$$

Next, for any DG function  $\Psi_h \in V_h$ , we define its continuous finite element projection  $\Psi_h^{FE} \in S_h$  by

$$(3.10) \quad \tilde{a}_h(\Psi_h^{FE}, v_h) = \tilde{a}_h(\Psi_h, v_h) \quad \forall v_h \in S_h,$$

where

$$\tilde{a}_h(u, v) = a_h(u, v) + \alpha(u, v),$$

and  $\alpha$  is a parameter that will be specified later in section 3.3.

A mesh-dependent  $H^{-1}$  norm will also be needed. To the end, we introduce the inverse discrete Laplace operator  $\Delta_h^{-1} : V_h \rightarrow \mathring{V}_h$  as follows: given  $\zeta \in V_h$ , let  $\Delta_h^{-1}\zeta \in \mathring{V}_h$  such that

$$(3.11) \quad a_h(-\Delta_h^{-1}\zeta, w_h) = (\zeta, w_h) \quad \forall w_h \in \mathring{V}_h.$$

We note that  $\Delta_h^{-1}$  is well defined provided that  $\sigma_e^0 > \sigma_*^0$  for some positive number  $\sigma_*^0$  and for all  $e \in \mathcal{E}_h$  because this condition ensures the coercivity of the DG bilinear form  $a_h(\cdot, \cdot)$ .

We then define “-1” inner product by

$$(3.12) \quad (\zeta, \xi)_{-1, h} := a_h(-\Delta_h^{-1}\zeta, -\Delta_h^{-1}\xi) = (\zeta, -\Delta_h^{-1}\xi) = (-\Delta_h^{-1}\zeta, \xi),$$

and the induced mesh-dependent  $H^{-1}$  norm is given by

$$(3.13) \quad \|\zeta\|_{-1, h} := \sqrt{(\zeta, \zeta)_{-1, h}} = \sup_{0 \neq \xi \in \mathring{V}_h} \frac{(\zeta, \xi)}{\|\xi\|_a},$$

where  $\|\xi\|_a := \sqrt{a_h(\xi, \xi)}$ . The following properties can be easily verified (cf. [3]):

$$(3.14) \quad |(\zeta, \xi)| \leq \|\zeta\|_{-1, h} \|\xi\|_a \quad \forall \xi \in V_h, \zeta \in \mathring{V}_h,$$

$$(3.15) \quad \|\zeta\|_{-1, h} \leq C\|\zeta\|_{L^2} \quad \forall \zeta \in \mathring{V}_h,$$

and, if  $\mathcal{T}_h$  is quasi-uniform, then

$$(3.16) \quad \|\zeta\|_{L^2} \leq Ch^{-1} \|\zeta\|_{-1, h} \quad \forall \zeta \in \mathring{V}_h.$$



**3.2. Discrete energy law and well-posedness.** In this subsection we first establish a discrete energy law, which mimics the differential energy law, for both fully discrete MIP-DG methods defined in (3.3)–(3.4). Based on this discrete energy law, we prove the existence and uniqueness of solutions to the MIP-DG methods by recasting the schemes as convex minimization problems at each time step. It turns out that the energy-splitting scheme is unconditionally stable but the fully implicit scheme is only conditionally stable.

**THEOREM 3.1.** *Let  $(U^m, W^m) \in V_h \times V_h$  be a solution to scheme (3.3)–(3.4). The following energy law holds for any  $h, k > 0$ :*

$$(3.17) \quad E_h(U^\ell) + k \sum_{m=1}^{\ell} \|d_t U^m\|_{-1,h}^2 + k^2 \sum_{m=1}^{\ell} \left\{ \frac{\epsilon}{2} \|d_t U^m\|_a^2 + \frac{1}{4\epsilon} \|d_t (U^m)^2\|_{L^2}^2 \right. \\ \left. + \frac{1}{2\epsilon} \|U^m d_t U^m\|_{L^2}^2 \pm \frac{1}{2\epsilon} \|d_t U^m\|_{L^2}^2 \right\} = E_h(U^0)$$

for all  $1 \leq \ell \leq M$ , where

$$(3.18) \quad E_h(U) := \frac{1}{4\epsilon} \|U^2 - 1\|_{L^2}^2 + \frac{\epsilon}{2} \|U\|_a^2.$$

Note that the sign “ $\pm$ ” in (3.17) takes “ $+$ ” when  $f^m = (U^m)^3 - U^{m-1}$  and “ $-$ ” when  $f^m = (U^m)^3 - U^m$ .

The proof of the above theorem follows from taking  $\eta = -\Delta^{-1} d_t U^m$  in (3.3) and  $v = d_t U^m$  in (3.4), adding the resulting two equations and combining like terms. We leave the detailed calculations to the interested reader.

**COROLLARY 3.2.** *Let  $\sigma_*^0 > 0$  be a sufficiently large constant. Suppose that  $\sigma_e^0 > \sigma_*^0$  for all  $e \in \mathcal{E}_h$ . Then scheme (3.3)–(3.4) is stable for all  $h, k > 0$  when  $f^m = (U^m)^3 - U^{m-1}$  and is stable for  $h > 0$  and  $k = O(\epsilon^3)$  when  $f^m = (U^m)^3 - U^m$ .*

*Proof.* The first case holds trivially from (3.17). In the second case, the “bad term”  $\|d_t U^m\|_{L^2}^2$  can be controlled by the “good terms”  $\|U^m\|_{-1,h}^2$  and  $\|U^m\|_a^2$  by using the norm interpolation inequality (3.40) provided that  $k = O(\epsilon^3)$ .  $\square$

**THEOREM 3.3.** *Suppose that  $\sigma_e^0 > \sigma_*^0$  for all  $e \in \mathcal{E}_h$ . Then scheme (3.3)–(3.4) has a unique solution  $(U^m, W^m)$  at each time step for all  $h, k > 0$  in the case  $f^m = (U^m)^3 - U^{m-1}$  and for  $h > 0$  and  $k = O(\epsilon^3)$  in the case  $f^m = (U^m)^3 - U^m$ .*

*Proof.* Setting  $\eta = -\Delta_h^{-1} v$  in (3.3) we get

$$(d_t U^m, v)_{-1,h} + (W^m, v) = 0.$$

Adding the above equation to (3.4) yields

$$(d_t U^m, v)_{-1,h} + \epsilon a_h(U^m, v) + \frac{1}{\epsilon} (f^m, v) = 0.$$

Hence,  $U^m$  satisfies

$$(3.19) \quad (U^m, v)_{-1,h} + k \epsilon a_h(U^m, v) + \frac{k}{\epsilon} (f^m, v) = (U^{m-1}, v)_{-1,h}.$$

In the case  $f^m = (U^m)^3 - U^{m-1}$  it is easy to check that (3.19) can be recast as a convex minimization problem (cf. [3, 13]) whose well-posedness holds for all  $h, k > 0$ . Hence, in this case there is a unique solution  $U^m$  to (3.3)–(3.4). On the

other hand, when  $f^m = (U^m)^3 - U^m$ , there is an extra term  $-k\epsilon^{-1}(U^m, v)$  comes out from the nonlinear term in (3.19). This extra term contributes a “bad term”  $-k\epsilon^{-1}\|U^m\|_{L^2}^2$  to the functional of the minimization problem. Again, this term can be controlled by the “good terms”  $\|U^m\|_{-1,h}^2$  and  $\|U^m\|_a^2$  in the functional by using the norm interpolation inequality (3.40), provided that  $k = O(\epsilon^3)$ . Hence, in the case  $f^m = (U^m)^3 - U^m$ , there is a unique solution  $U^m$  to (3.3)–(3.4) for all  $h > 0$  and  $k = O(\epsilon^3)$ . The proof is complete.  $\square$

**3.3. Discrete spectrum estimate on the DG space.** In this subsection, we shall establish a discrete spectrum estimate for the linearized Cahn-Hilliard operator on the DG space, which plays a vital role in our error estimates.

To the end, we first state a slightly modified version of a discrete spectrum estimate for the linearized Cahn-Hilliard operator on the continuous finite element space first proved in [14, 15]. Due to the close similarity, we omit the proof of this modified version and refer the interested reader to [14, 15].

LEMMA 3.4. *Suppose the assumptions of Lemma 2.2 hold, and  $C_0$  is the same as in (2.22).  $C_1$  and  $C_2$  are defined by*

$$(3.20) \quad C_1 := \max_{|\xi| \leq 2C_0} |f''(\xi)|,$$

$$(3.21) \quad \|u - \hat{P}_h u\|_{L^\infty((0,T);L^\infty)} \leq C_2 h^{2-\frac{d}{2}} \epsilon^{\min\{-\sigma_1-\frac{5}{2}, -\sigma_3-1\}}.$$

Then there exists  $0 < \epsilon_1 \ll 1$  such that, for any  $\epsilon \in (0, \epsilon_1)$ , there holds

$$(3.22) \quad \lambda_{CH}^{FE} \equiv \inf_{0 \neq \psi_h \in L_0^2(\Omega) \cap S_h} \frac{\epsilon \|\nabla \psi_h\|_{L^2}^2 + \frac{2-\epsilon^3}{2\epsilon} (f'(\hat{P}_h u) \psi_h, \psi_h)}{\|\nabla \Delta^{-1} \psi_h\|_{L^2}^2} \geq -(C_0 + 1),$$

provided that  $h$  satisfies

$$(3.23) \quad h^{2-\frac{d}{2}} \leq (C_1 C_2)^{-1} \epsilon^{\max\{\sigma_1+\frac{11}{2}, \sigma_3+4\}}.$$

Here  $\Delta^{-1} : L_0^2(\Omega) \rightarrow H^1(\Omega) \cap L_0^2(\Omega)$  denotes the inverse Laplace operator.

We are now ready to state the discrete spectrum estimate on the DG space.

PROPOSITION 3.5. *Suppose the assumptions of Lemma 2.2 hold. Let  $u$  be the solution of (1.1)–(1.5) and  $P_h u$  denote its DG elliptic projection. Assume*

$$(3.24) \quad \operatorname{ess\,sup}_{t \in [0, \infty)} \|u\|_{W^{1+r, \infty}} \leq C \epsilon^{-\gamma},$$

for a constant  $\gamma$ , then there exists  $0 < \epsilon_2 \ll 1$  and an  $\epsilon$ -independent and  $h$ -independent constant  $c_0 > 0$ , such that for any  $\epsilon \in (0, \epsilon_2)$ , there holds

$$(3.25) \quad \lambda_{CH}^{DG} = \inf_{0 \neq \Phi_h \in L_0^2(\Omega) \cap V_h} \frac{\epsilon a_h(\Phi_h, \Phi_h) + \frac{1-\epsilon^3}{\epsilon} (f'(P_h u) \Phi_h, \Phi_h)}{\|\nabla \Delta^{-1} \Phi_h\|_{L^2}^2} \geq -c_0,$$

provided that  $h$  satisfies the constraints

$$(3.26) \quad h^{2-\frac{d}{2}} \leq (C_1 C_2)^{-1} \epsilon^{\max\{\sigma_1+\frac{11}{2}, \sigma_3+4\}},$$

$$(3.27) \quad h^{1+r} |\ln h|^{\bar{r}} \leq (C_1 C_3)^{-1} \epsilon^{\gamma+3}.$$

where  $C_1$  and  $C_2$  are same as in Lemma 3.4,  $\bar{r}$  and  $C_3$  are defined by

$$\bar{r} = \min\{1, r\} - \min\{1, r-1\},$$

$$\|u - P_h u\|_{L^\infty((0,T);L^\infty)} \leq C_3 h^{1+r} |\ln h|^{\bar{r}} \epsilon^{-\gamma}.$$

*Proof.* By Proposition 2 in [14], under the mesh constraint (3.26), we have

$$(3.28) \quad \|f'(\widehat{P}_h u) - f'(u)\|_{L^\infty((0,T);L^\infty)} \leq \epsilon^3.$$

Similarly, under the mesh condition (3.27), we can show that for any  $\epsilon > 0$ , there holds

$$(3.29) \quad \|f'(P_h u) - f'(u)\|_{L^\infty((0,T);L^\infty)} \leq \epsilon^3.$$

It follows from (3.28) and (3.29) that

$$(3.30) \quad \|f'(P_h u) - f'(\widehat{P}_h u)\|_{L^\infty((0,T);L^\infty)} \leq 2\epsilon^3 \quad \text{and} \quad f'(P_h u) \geq f'(\widehat{P}_h u) - 2\epsilon^3.$$

Therefore,

$$(3.31) \quad \begin{aligned} & \epsilon a_h(\Phi_h, \Phi_h) + \frac{1 - \epsilon^3}{\epsilon} (f'(P_h u) \Phi_h, \Phi_h) \\ & \geq \epsilon a_h(\Phi_h, \Phi_h) + \frac{1 - \epsilon^3}{\epsilon} (f'(\widehat{P}_h u) \Phi_h, \Phi_h) - 2\epsilon^2(1 - \epsilon^3) \|\Phi_h\|_{L^2}^2 \\ & = \epsilon \frac{1 - \epsilon^3}{1 - \frac{\epsilon^3}{2}} a_h(\Phi_h, \Phi_h) + \frac{1 - \epsilon^3}{\epsilon} (f'(\widehat{P}_h u) \Phi_h, \Phi_h) \\ & \quad - 2\epsilon^2(1 - \epsilon^3) \|\Phi_h\|_{L^2}^2 + \frac{\epsilon^4}{2 - \epsilon^3} a_h(\Phi_h, \Phi_h) \\ & = \epsilon \frac{1 - \epsilon^3}{1 - \frac{\epsilon^3}{2}} a_h(\Phi_h, \Phi_h) + \frac{1 - \epsilon^3}{\epsilon} \int_{\Omega} f'(\widehat{P}_h u) \left( (\Phi_h)^2 - (\Phi_h^{FE})^2 \right) dx \\ & \quad + \frac{1 - \epsilon^3}{\epsilon} \int_{\Omega} f'(\widehat{P}_h u) (\Phi_h^{FE})^2 dx - 2\epsilon^2(1 - \epsilon^3) \|\Phi_h\|_{L^2}^2 + \frac{\epsilon^4}{2 - \epsilon^3} a_h(\Phi_h, \Phi_h). \end{aligned}$$

Next, we derive a lower bound for each of the first two terms on the right-hand side of (3.31). Notice that the first term can be rewritten as

$$(3.32) \quad \begin{aligned} a_h(\Phi_h, \Phi_h) &= a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) + 2a_h(\Phi_h, \Phi_h^{FE}) - a_h(\Phi_h^{FE}, \Phi_h^{FE}) \\ &= a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) + \|\nabla \Phi_h^{FE}\|_{L^2}^2 + 2\alpha \|\Phi_h^{FE} - \Phi_h\|_{L^2}^2 \\ & \quad + 2\alpha (\Phi_h^{FE} - \Phi_h, \Phi_h). \end{aligned}$$

To bound  $\|\Phi_h - \Phi_h^{FE}\|_{L^2}$  from above, we consider the following auxiliary problem:

$$\widetilde{a}_h(\phi, \chi) = (\Phi_h - \Phi_h^{FE}, \chi) \quad \forall \chi \in H^1(\Omega).$$

For  $\sigma_e^0 > \sigma_*^0$  for all  $e \in \mathcal{E}_h$ , the above problem has a unique solution  $\phi \in H^{1+\theta}(\Omega)$  for  $0 < \theta \leq 1$  such that

$$(3.33) \quad \|\phi\|_{H^{1+\theta}(\Omega)} \leq C \|\Phi_h - \Phi_h^{FE}\|_{L^2} \quad \text{for } \theta \in (0, 1].$$

By the definition of  $\Phi_h^{FE}$ , we immediately get the following Galerkin orthogonality:

$$\widetilde{a}_h(\Phi_h - \Phi_h^{FE}, \chi_h) = 0 \quad \forall \chi_h \in S_h,$$

It follows from the duality argument (cf. [21, Theorem 2.14]) that

$$(3.34) \quad \begin{aligned} \|\Phi_h - \Phi_h^{FE}\|_{L^2}^2 &\leq Ch^{2\theta} \widetilde{a}_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) \\ &\leq Ch^{2\theta} a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) + Ch^{2\theta} \alpha \|\Phi_h - \Phi_h^{FE}\|_{L^2}^2. \end{aligned}$$

For all  $h$  satisfying  $Ch^{2\theta}\alpha < 1$ , we get

$$(3.35) \quad \|\Phi_h - \Phi_h^{FE}\|_{L^2}^2 \leq \frac{Ch^{2\theta}}{1 - Ch^{2\theta}\alpha} a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}).$$

Now the last term on the right-hand side of (3.32) can be bounded as follows:

$$(3.36) \quad \begin{aligned} 2\alpha(\Phi_h^{FE} - \Phi_h, \Phi_h) &\geq -2\alpha\|\Phi_h^{FE} - \Phi_h\|_{L^2}\|\Phi_h\|_{L^2} \\ &\geq -2\alpha\sqrt{\frac{Ch^{2\theta}a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE})}{1 - Ch^{2\theta}\alpha}}\|\Phi_h\|_{L^2} \\ &\geq -\frac{1}{2}a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) - \frac{2C\alpha^2h^{2\theta}}{1 - Ch^{2\theta}\alpha}\|\Phi_h\|_{L^2}^2. \end{aligned}$$

The second term on the right-hand side of (3.31) can be bounded by

$$(3.37) \quad \begin{aligned} \int_{\Omega} f'(\hat{P}_h u)((\Phi_h)^2 - (\Phi_h^{FE})^2) dx &\geq -C \int_{\Omega} |(\Phi_h)^2 - (\Phi_h^{FE})^2| dx \\ &= -C \int_{\Omega} |-(\Phi_h - \Phi_h^{FE})^2 + 2\Phi_h(\Phi_h - \Phi_h^{FE})| dx \\ &\geq -C\|\Phi_h - \Phi_h^{FE}\|_{L^2}^2 - \frac{\epsilon^3(1 - \epsilon^3)}{1 - \frac{\epsilon^3}{2}}\|\Phi_h\|_{L^2}^2 - C\frac{1 - \frac{\epsilon^3}{2}}{\epsilon^3(1 - \epsilon^3)}\|\Phi_h - \Phi_h^{FE}\|_{L^2}^2. \end{aligned}$$

Here we have used the facts that

$$(3.38) \quad \|u\|_{L^\infty((0,T);L^\infty)} \leq C, \quad |f'(\hat{P}_h u)| \leq |f'(u)| + \epsilon^3 \leq C.$$

Substituting (3.35) into (3.37) yields

$$(3.39) \quad \begin{aligned} \frac{1 - \epsilon^3}{\epsilon} \int_{\Omega} f'(\hat{P}_h u)((\Phi_h)^2 - (\Phi_h^{FE})^2) dx \\ \geq -\gamma_3 \frac{\epsilon(1 - \epsilon^3)}{1 - \frac{\epsilon^3}{2}} a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) - \frac{\epsilon^2(1 - \epsilon^3)}{1 - \frac{\epsilon^3}{2}} \|\Phi_h\|_{L^2}^2, \end{aligned}$$

where

$$\gamma_3 \geq \frac{Ch^{2\theta}}{1 - Ch^{2\theta}\alpha} \cdot 2C \frac{1 - \frac{\epsilon^3}{2}}{\epsilon(1 - \epsilon^3)} \left(1 + \frac{1 - \frac{\epsilon^3}{2}}{\epsilon^3(1 - \epsilon^3)}\right),$$

and  $h$  is chosen small enough such that  $\gamma_3 < 1/4$ .

The term  $\|\Phi_h\|_{L^2}^2$  can be bounded by

$$(3.40) \quad \begin{aligned} \|\Phi_h\|_{L^2}^2 = (\Phi_h, \Phi_h) &= a_h(\Delta_h^{-1}\Phi_h, \Phi_h) \leq a_h(\Delta_h^{-1}\Phi_h, \Delta_h^{-1}\Phi_h)^{\frac{1}{2}} a_h(\Phi_h, \Phi_h)^{\frac{1}{2}} \\ &\leq \frac{\rho}{2} a_h(\Delta_h^{-1}\Phi_h, \Delta_h^{-1}\Phi_h) + \frac{1}{2\rho} a_h(\Phi_h, \Phi_h) \end{aligned}$$

for any constant  $\rho > 0$ .

Adding the fifth term on the right-hand side of (3.31), the last term on the right-hand side of (3.36) and that of (3.39), we get for all  $h$  satisfying  $2C\alpha^2h^{2\theta}/(1 - Ch^{2\theta}\alpha) \leq \epsilon$

$$(3.41) \quad \begin{aligned} -\left(\frac{\epsilon(1 - \epsilon^3)}{1 - \frac{\epsilon^3}{2}} \frac{2C\alpha^2h^{2\theta}}{1 - Ch^{2\theta}\alpha} + \frac{3\epsilon^2(1 - \epsilon^3)}{1 - \frac{\epsilon^3}{2}}\right) \|\Phi_h\|_{L^2}^2 &\geq -\frac{4\epsilon^2(1 - \epsilon^3)}{1 - \frac{\epsilon^3}{2}} \|\Phi_h\|_{L^2}^2 \\ &\geq -\frac{\epsilon^4}{2(2 - \epsilon^3)} a_h(\Phi_h, \Phi_h) - C a_h(\Delta_h^{-1}\Phi_h, \Delta_h^{-1}\Phi_h). \end{aligned}$$

Combining (3.32), (3.36), (3.39) and (3.41) with (3.31), we have

$$\begin{aligned}
(3.42) \quad & \epsilon a_h(\Phi_h, \Phi_h) + \frac{1-\epsilon^3}{\epsilon} \int_{\Omega} f'(P_h u)(\Phi_h)^2 dx \\
& \geq \frac{\epsilon(1-\epsilon^3)}{4-2\epsilon^3} a_h(\Phi_h - \Phi_h^{FE}, \Phi_h - \Phi_h^{FE}) + \frac{2\alpha\epsilon(1-\epsilon^3)}{1-\frac{\epsilon^3}{2}} \|\Phi_h^{FE} - \Phi_h\|_{L^2}^2 \\
& + \frac{\epsilon(1-\epsilon^3)}{1-\frac{\epsilon^3}{2}} \|\nabla \Phi_h^{FE}\|_{L^2}^2 - C a_h(\Delta_h^{-1} \Phi_h, \Delta_h^{-1} \Phi_h) \\
& + \frac{1-\epsilon^3}{\epsilon} \int_{\Omega} f'(\widehat{P}_h u)(\Phi_h^{FE})^2 dx + \frac{\epsilon^4}{2(2-\epsilon^3)} a_h(\Phi_h, \Phi_h).
\end{aligned}$$

Applying the spectrum estimate (3.22), we get

$$\begin{aligned}
& \epsilon \frac{1-\epsilon^3}{1-\frac{\epsilon^3}{2}} \|\nabla \Phi_h^{FE}\|_{L^2}^2 + \frac{1-\epsilon^3}{\epsilon} \int_{\Omega} f'(\widehat{P}_h u)(\Phi_h^{FE})^2 dx \\
& = \frac{1-\epsilon^3}{1-\frac{\epsilon^3}{2}} \left( \epsilon \|\nabla \Phi_h^{FE}\|_{L^2}^2 + \frac{1-\frac{\epsilon^3}{2}}{\epsilon} \int_{\Omega} f'(\widehat{P}_h u)(\Phi_h^{FE})^2 dx \right) \\
& \geq -\frac{1-\epsilon^3}{1-\frac{\epsilon^3}{2}} (C_0 + 1) \|\nabla \Delta^{-1} \Phi_h^{FE}\|_{L^2}^2,
\end{aligned}$$

which together with (3.42) implies that

$$\begin{aligned}
(3.43) \quad & \epsilon a_h(\Phi_h, \Phi_h) + \frac{1-\epsilon^3}{\epsilon} \int_{\Omega} f'(P_h u)(\Phi_h)^2 dx \\
& \geq -C a_h(\Delta_h^{-1} \Phi_h, \Delta_h^{-1} \Phi_h) - C \|\nabla \Delta^{-1} \Phi_h^{FE}\|_{L^2}^2 + \frac{2\alpha\epsilon(1-\epsilon^3)}{1-\frac{\epsilon^3}{2}} \|\Phi_h^{FE} - \Phi_h\|_{L^2}^2.
\end{aligned}$$

By the stability of  $\Delta^{-1}$ , we have

$$\|\nabla \Delta^{-1}(\Phi_h - \Phi_h^{FE})\|_{L^2}^2 \leq \widehat{C} \|\Phi_h - \Phi_h^{FE}\|_{L^2}^2,$$

which together with the triangle inequality yields

$$\|\nabla \Delta^{-1} \Phi_h^{FE}\|_{L^2}^2 \leq 2 \|\nabla \Delta^{-1} \Phi_h\|_{L^2}^2 + 2\widehat{C} \|\Phi_h - \Phi_h^{FE}\|_{L^2}^2.$$

Similarly, since  $\Delta_h^{-1} \Phi_h$  is the elliptic projection of  $\Delta^{-1} \Phi_h$ , there holds

$$a_h(\Delta_h^{-1} \Phi_h, \Delta_h^{-1} \Phi_h) \leq C \|\nabla \Delta^{-1} \Phi_h\|_{L^2}^2.$$

Therefore, choosing  $\alpha = O(\widehat{C}\epsilon^{-1})$ , (3.43) can be further reduced into

$$\epsilon a_h(\Phi_h, \Phi_h) + \frac{1-\epsilon^3}{\epsilon} \int_{\Omega} f'(P_h u)(\Phi_h)^2 dx \geq -c_0 \|\nabla \Delta^{-1} \Phi_h\|_{L^2}^2$$

for some  $c_0 > 0$ . This proves (3.25), and the proof is complete.  $\square$

**3.4. Error analysis.** In this subsection, we shall derive some optimal error estimates for the proposed MIP-DG schemes (3.3)–(3.4), in which the constants in the error bounds depend on  $\epsilon^{-1}$  only in low polynomial orders, instead of exponential orders. The key to obtaining such refined error bounds is to use the discrete spectrum estimate (3.25). In addition, the nonlinear Gronwall inequality presented in Lemma 2.3 also plays an important role in the proof. To ease the presentation, we set  $r = 1$  in this subsection and section 4, and generalization to  $r > 1$  can be proven similarly.

The main results of this subsection are stated in the following theorem.

**THEOREM 3.6.** *Let  $\{(U^m, W^m)\}_{m=0}^M$  be the solution of scheme (3.3)–(3.4) with  $r = 1$ . Suppose that (GA) holds and  $\sigma_e^0 > \sigma_*^0$  for all  $e \in \mathcal{E}_h$ , and define*

$$(3.44) \quad \rho_3(\epsilon) := \epsilon^{-\max\{2\sigma_1 + \frac{13}{2}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} - 4},$$

$$(3.45) \quad r(h, k; \epsilon, d, \sigma_i) := k^2 \rho_1(\epsilon; d) + h^6 \rho_3(\epsilon).$$

Then, under the following mesh and starting value conditions:

$$(3.46) \quad h^{2-\frac{d}{2}} \leq (C_1 C_2)^{-1} \epsilon^{\max\{\sigma_1 + \frac{11}{2}, \sigma_3 + 4\}},$$

$$(3.47) \quad h^{1+r} |\ln h|^{\bar{r}} \leq (C_1 C_3)^{-1} \epsilon^{\gamma+3},$$

$$(3.48) \quad k \leq \epsilon^3 \quad \text{when } f^m = (U^m)^3 - U^m,$$

$$(3.49) \quad h^{2\theta} \leq C \frac{\epsilon(1-\epsilon^3)}{8-4\epsilon^3},$$

$$(3.50) \quad k \leq C \epsilon^{\frac{4(6+d)}{4-d} + (4d-2)\sigma_1},$$

$$(3.51) \quad (U^0, 1) = (u_0, 1),$$

$$(3.52) \quad \|u_0 - U^0\|_{H^{-1}} \leq C h^3 \|u_0\|_{H^2},$$

there hold the error estimates

$$(3.53) \quad \max_{0 \leq m \leq M} \|u(t_m) - U^m\|_{-1,h} + \left( \sum_{m=1}^M k^2 \|d_t(u(t_m) - U^m)\|_{-1,h}^2 \right)^{\frac{1}{2}} \\ \leq C r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}},$$

$$(3.54) \quad \left( k \sum_{m=1}^M \|u(t_m) - U^m\|_{L^2}^2 \right)^{\frac{1}{2}} \\ \leq C \left( h^2 \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\}} + \epsilon^{-2} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} \right),$$

$$(3.55) \quad \left( k \sum_{m=1}^M \|\nabla(u(t_m) - U^m)\|_{L^2}^2 \right)^{\frac{1}{2}} \\ \leq C \left( h \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\}} + \epsilon^{-2} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} \right).$$

Moreover, if the starting value  $U^0$  satisfies

$$(3.56) \quad \|u_0 - U^0\|_{L^2} \leq C h^2 \|u_0\|_{H^2},$$

then there hold

$$(3.57) \quad \max_{0 \leq m \leq M} \|u(t_m) - U^m\|_{L^2} + \left( k \sum_{m=1}^M k \|d_t(u(t_m) - U^m)\|_{L^2}^2 \right)^{\frac{1}{2}} \\ + \left( \frac{k}{\epsilon} \sum_{m=1}^M \|w(t_m) - W^m\|_{L^2}^2 \right)^{\frac{1}{2}} \\ \leq C \left( h^2 \rho_3(\epsilon)^{\frac{1}{2}} + \epsilon^{-\frac{7}{2}} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} \right)$$

$$(3.58) \quad \max_{0 \leq m \leq M} \|u(t_m) - U^m\|_{L^\infty} \\ \leq C \left( h^2 |\ln h| \epsilon^{-\gamma} + h^{-\frac{d}{2}} \epsilon^{-\frac{7}{2}} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} \right).$$

Furthermore, suppose that the starting value  $W^0$  satisfies

$$(3.59) \quad \|P_h w_0 - W^0\|_{L^2} \leq C h^\beta$$

for some  $\beta > 1$ , and there exists a constant  $\gamma'$  such that

$$(3.60) \quad \operatorname{ess\,sup}_{t \in [0, \infty)} \|w\|_{W^{2, \infty}} \leq C \epsilon^{-\gamma'},$$

then we have

$$(3.61) \quad \max_{0 \leq m \leq M} \|w(t_m) - W^m\|_{L^2} \leq C \left( h^2 \rho_3(\epsilon) + h^\beta \right. \\ \left. + k^{-\frac{1}{2}} \epsilon^{-3} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} \right),$$

$$(3.62) \quad \max_{0 \leq m \leq M} \|w(t_m) - W^m\|_{L^\infty} \leq C \left( h^{-\frac{d}{2}} \left( k^{-\frac{1}{2}} \epsilon^{-3} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} + h^\beta \right) \right. \\ \left. + h^2 |\ln h| \epsilon^{-\gamma'} \right).$$

*Proof.* In the following, we only give a proof for the convex splitting scheme corresponding to  $f^m = (u^m)^3 - u^{m-1}$  in (3.13) because the proof for the fully implicit scheme with  $f^m = (u^m)^3 - u^m$  is almost same. Since the proof is long, we divide it into four steps.

*Step 1:* It is obvious that equations (1.1)–(1.4) imply that

$$(3.63) \quad (u_t(t_m), \eta_h) + a_h(w(t_m), \eta_h) = 0 \quad \forall \eta_h \in V_h,$$

$$(3.64) \quad \epsilon a_h(u(t_m), v_h) + \frac{1}{\epsilon} (f(u(t_m)), v_h) = (w(t_m), v_h) \quad \forall v_h \in V_h.$$

Define error functions  $E^m := u(t_m) - U^m$  and  $G^m := w(t_m) - W^m$ . Subtracting (3.3) from (3.63) and (3.4) from (3.64) yield the following error equations:

$$(3.65) \quad (d_t E^m, \eta_h) + a_h(G^m, \eta_h) = (R(u_{tt}, m), \eta_h) \quad \forall \eta_h \in V_h,$$

$$(3.66) \quad \epsilon a_h(E^m, v_h) + \frac{1}{\epsilon} (f(u(t_m)) - f(U^m), v_h) = (G^m, v_h) \quad \forall v_h \in V_h,$$

where

$$R(u_{tt}; m) := \frac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1}) u_{tt}(s) ds.$$

It follows from (2.19) that

$$\begin{aligned} k \sum_{m=1}^M \|R(u_{tt}; m)\|_{H^{-1}}^2 &\leq \frac{1}{k} \sum_{m=1}^M \left( \int_{t_{m-1}}^{t_m} (s - t_{m-1})^2 ds \right) \left( \int_{t_{m-1}}^{t_m} \|u_{tt}(s)\|_{H^{-1}}^2 ds \right) \\ &\leq Ck^2 \rho_1(\epsilon, d). \end{aligned}$$

Introduce the error decompositions

$$(3.67) \quad E^m = \Theta^m + \Phi^m, \quad G^m = \Lambda^m + \Psi^m,$$

where

$$\begin{aligned} \Theta^m &:= u(t_m) - P_h u(t_m), & \Phi^m &:= P_h u(t_m) - U^m, \\ \Lambda^m &:= w(t_m) - P_h w(t_m), & \Psi^m &:= P_h w(t_m) - W^m. \end{aligned}$$

Using the definition of the operator  $P_h$  in (3.6), (3.65)–(3.66) can be rewritten as

$$(3.68) \quad (d_t \Phi^m, \eta_h) + a_h(\Psi^m, \eta_h) = -(d_t \Theta^m, \eta_h) + (R(u_{tt}, m), \eta_h) \quad \forall \eta_h \in V_h,$$

$$(3.69) \quad \epsilon a_h(\Phi^m, v_h) + \frac{1}{\epsilon} (f(u(t_m)) - f^m, v_h) = (\Psi^m, v_h) + (\Lambda^m, v_h) \quad \forall v_h \in V_h.$$

Setting  $\eta_h = -\Delta_h^{-1} \Phi^m$  in (3.68) and  $v_h = \Phi^m$  in (3.69), adding the resulting equations and summing over  $m$  from 1 to  $\ell$ , we get

$$\begin{aligned} (3.70) \quad &a_h(\Delta_h^{-1} \Phi^\ell, \Delta_h^{-1} \Phi^\ell) + \sum_{m=1}^{\ell} a_h(\Delta_h^{-1} \Phi^m - \Delta_h^{-1} \Phi^{m-1}, \Delta_h^{-1} \Phi^m - \Delta_h^{-1} \Phi^{m-1}) \\ &+ 2k \sum_{m=1}^{\ell} \epsilon a_h(\Phi^m, \Phi^m) + 2k \sum_{m=1}^{\ell} \frac{1}{\epsilon} (f(u(t_m)) - f^m, \Phi^m) \\ &= 2k \sum_{m=1}^{\ell} \left( (R(u_{tt}, m), -\Delta_h^{-1} \Phi^m) - (d_t \Theta^m, -\Delta_h^{-1} \Phi^m) + (\Lambda^m, \Phi^m) \right) \\ &\quad + a_h(\Delta_h^{-1} \Phi^0, \Delta_h^{-1} \Phi^0). \end{aligned}$$

*Step 2:* For  $\sigma_e^0 > \sigma_*^e$  for all  $e \in \mathcal{E}_h$ , the first long term on the right-hand side of (3.70) can be bounded as follows

$$\begin{aligned} (3.71) \quad &2k \sum_{m=1}^{\ell} \left( (R(u_{tt}, m), -\Delta_h^{-1} \Phi^m) + (d_t \Theta^m, -\Delta_h^{-1} \Phi^m) + (\Lambda^m, \Phi^m) \right) \\ &\leq Ck \sum_{m=1}^{\ell} \left( \|R(u_{tt}; m)\|_{H^{-1}}^2 + \|d_t \Theta^m\|_{H^{-1}}^2 + (1 - \epsilon^3) \epsilon^{-4} \|\Lambda^m\|_{H^{-1}}^2 \right) \\ &\quad + k \sum_{m=1}^{\ell} \left( a_h(\Delta_h^{-1} \Phi^m, \Delta_h^{-1} \Phi^m) + \frac{\epsilon^4}{1 - \epsilon^3} a_h(\Phi^m, \Phi^m) \right) \\ &\leq k \sum_{m=1}^{\ell} \left( a_h(\Delta_h^{-1} \Phi^m, \Delta_h^{-1} \Phi^m) + \frac{\epsilon^4}{1 - \epsilon^3} a_h(\Phi^m, \Phi^m) \right) \\ &\quad + C \left( k^2 \rho_1(\epsilon, d) + h^6 \rho_3(\epsilon) \right), \end{aligned}$$



where we have used (2.15) and the following facts [10]:

$$\|u - P_h u\|_{H^{-1}} \leq Ch^3 \|u\|_{H^2}, \quad \|w - P_h w\|_{H^{-1}} \leq Ch^3 \|w\|_{H^2}.$$

We now bound the last term on the left-hand side of (3.70). By the definition of  $f^m$ , we have

$$\begin{aligned} f(u(t_m)) - f^m &= f(u(t_m)) - f(P_h u(t_m)) + f(P_h u(t_m)) - f^m \\ &\geq -|f(u(t_m)) - f(P_h u(t_m))| + (P_h u(t_m))^3 - P_h u(t_m) - (U^m)^3 + U^{m-1} \\ &\geq -C|\Theta^m| + \left((P_h u(t_m))^2 + P_h u(t_m) U^m + (U^m)^2\right) \Phi^m - \Phi^m - k d_t U^m \\ &\geq -C|\Theta^m| + f'(P_h u(t_m)) \Phi^m - 3P_h u(t_m) (\Phi^m)^2 + (\Phi^m)^3 - k d_t U^m. \end{aligned}$$

By the discrete energy law (3.17), (3.13) and (3.40), we obtain for any  $1 \leq \ell \leq M$

$$\begin{aligned} (3.72) \quad & 2k \sum_{m=1}^{\ell} \frac{1}{\epsilon} (f(u(t_m)) - f^m, \Phi^m) \\ & \geq -\frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Theta^m\|_{H^{-1}(\mathcal{T}_h)} \|\Phi^m\|_{H^1(\mathcal{T}_h)} + 2k \sum_{m=1}^{\ell} \frac{1}{\epsilon} \left( f'(P_h u(t_m)), (\Phi^m)^2 \right) \\ & \quad - \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^3}^3 + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 - \frac{2k}{\epsilon} \sum_{m=1}^{\ell} k \|d_t U^m\|_{-1,h} \|\Phi^m\|_{\alpha} \\ & \geq 2k \sum_{m=1}^{\ell} \frac{1}{\epsilon} \left( f'(P_h u(t_m)), (\Phi^m)^2 \right) + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 - \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^3}^3 \\ & \quad - k\epsilon^4 \sum_{m=1}^{\ell} \|\Phi^m\|_a^2 - C \left( h^6 \epsilon^{-6} \|u\|_{L^2((0,T);H^s(\Omega))}^2 + k^2 \epsilon^{-6} E_h(u_h^0) \right) \\ & \geq 2k \sum_{m=1}^{\ell} \frac{1}{\epsilon} \left( f'(P_h u(t_m)), (\Phi^m)^2 \right) + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 - \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^3}^3 \\ & \quad - k \frac{\epsilon^4}{1 - \epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) - C \left( h^6 \epsilon^{-6} \|u\|_{L^2((0,T);H^s(\Omega))}^2 + k^2 \epsilon^{-6} E_h(U^0) \right). \end{aligned}$$

Substituting (3.71) and (3.72) into (3.70) we get

$$\begin{aligned}
(3.73) \quad & a_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell) + \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}, \Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}) \\
& + \frac{2k(1-5\epsilon^3)}{1-\epsilon^3} \sum_{m=1}^{\ell} \left( \epsilon a_h(\Phi^m, \Phi^m) + \frac{1-\epsilon^3}{\epsilon} (f'(P_h u(t_m))\Phi^m, \Phi^m) \right) \\
& + \frac{6\epsilon^4}{1-\epsilon^3} k \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 \\
& \leq Ck \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) + \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^3}^3 \\
& \quad - 10k\epsilon^2 \sum_{m=1}^{\ell} (f'(P_h u(t_m))\Phi^m, \Phi^m) + C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)) \\
& \quad + C\left(h^6\epsilon^{-6}\|u\|_{L^2((0,T);H^s(\Omega))}^2 + k^2\epsilon^{-6}E_h(U^0)\right).
\end{aligned}$$

*Step 3:* To control the second term on the right-hand side of (3.73), we appeal to the following Gagliardo-Nirenberg inequality [1]:

$$\|v\|_{L^3(K)}^3 \leq C\left(\|\nabla v\|_{L^2(K)}^{\frac{d}{2}}\|v\|_{L^2(K)}^{\frac{6-d}{2}} + \|v\|_{L^2(K)}^3\right) \quad \forall K \in \mathcal{T}_h.$$

Thus we get

$$\begin{aligned}
(3.74) \quad & \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^3}^3 \leq \epsilon^4 k \sum_{m=1}^{\ell} \|\nabla \Phi^m\|_{L^2(\mathcal{T}_h)}^2 + \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^2}^3 \\
& \quad + C\epsilon^{-\frac{4(1+d)}{4-d}} k \sum_{m=1}^{\ell} \|\Phi^m\|_{L^2}^{\frac{2(6-d)}{4-d}} \\
& \leq \frac{\epsilon^4}{1-\epsilon^3} k \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^2}^3 \\
& \quad + C\epsilon^{-\frac{4(1+d)}{4-d}} k \sum_{m=1}^{\ell} \|\Phi^m\|_{L^2}^{\frac{2(6-d)}{4-d}}.
\end{aligned}$$

The third item on the right-hand side of (3.73) can be bounded by

$$\begin{aligned}
(3.75) \quad & -10k\epsilon^2 (f'(P_h u(t_m))\Phi^m, \Phi^m) \\
& \leq k \frac{\epsilon^4}{1-\epsilon^3} a_h(\Phi^m, \Phi^m) + kC a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m).
\end{aligned}$$

Again, here we have used (3.40).

Finally, for the third term on the left-hand side of (3.73), we utilize the discrete spectrum estimate (3.25) to bound it from below as follows:

$$(3.76) \quad \epsilon a_h(\Phi^m, \Phi^m) + \frac{1-\epsilon^3}{\epsilon} (f'(P_h u(t_m))\Phi^m, \Phi^m) \geq -c_0 \|\nabla \Delta^{-1}\Phi^m\|_{L^2}^2.$$

By the stability of  $\Delta^{-1}$  and (3.40), we also have

$$(3.77) \quad c_0 \|\nabla \Delta^{-1} \Phi^m\|_{L^2}^2 \leq C \|\Phi^m\|_{L^2}^2 \leq \frac{\epsilon^4}{1-\epsilon^3} a_h(\Phi^m, \Phi^m) + C a_h(\Delta_h^{-1} \Phi^m, \Delta_h^{-1} \Phi^m).$$

*Step 4:* Substituting (3.74), (3.75), (3.76), (3.77) into (3.73), we get

$$(3.78) \quad \begin{aligned} & a_h(\Delta_h^{-1} \Phi^\ell, \Delta_h^{-1} \Phi^\ell) + \sum_{m=1}^{\ell} a_h(\Delta_h^{-1} \Phi^m - \Delta_h^{-1} \Phi^{m-1}, \Delta_h^{-1} \Phi^m - \Delta_h^{-1} \Phi^{m-1}) \\ & + \frac{2\epsilon^4 k}{1-\epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 \\ & \leq Ck \sum_{m=1}^{\ell} a_h(\Delta_h^{-1} \Phi^m, \Delta_h^{-1} \Phi^m) + \frac{Ck}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^2}^3 \\ & + C\epsilon^{-\frac{4(1+d)}{4-d}} k \sum_{m=1}^{\ell} \|\Phi^m\|_{L^2}^{\frac{2(6-d)}{4-d}} + C(k^2 \rho_1(\epsilon; d) + h^6 \rho_3(\epsilon)) \\ & + C(h^6 \epsilon^{-6} \|u\|_{L^2((0,T);H^s(\Omega))}^2 + k^2 \epsilon^{-6} E_h(U^0)). \end{aligned}$$

By discrete energy law (3.17), General Assumption (2.3),  $H^1$  stability of elliptic projection,  $L^\infty$  stability(or  $L^\infty$  error estimate and triangle inequality) of elliptic projection, we can get for any  $0 \leq \ell \leq M$

$$\|U^\ell\|_{L^2} \leq k \sum_{m=1}^{\ell} \|d_t U^m\|_{L^2} + \|U^0\|_{L^2} \leq C\epsilon^{-\sigma_1}.$$

Since the projection of  $u$  is bounded, then for any  $0 \leq \ell \leq M$

$$(3.79) \quad \|\Phi^\ell\|_{L^2} \leq C\epsilon^{-\sigma_1}.$$

We point out that the exponent for  $\|\Phi^m\|_{L^2}$  is  $\frac{2(6-d)}{4-d}$ , which is bigger than 3 for  $d = 2, 3$ . By (3.79) we have

$$\|\Phi^m\|_{L^2}^4 \leq C\epsilon^{-\sigma_1} \|\Phi^m\|_{L^2}^3, \quad \|\Phi^m\|_{L^2}^6 \leq C\epsilon^{-3\sigma_1} \|\Phi^m\|_{L^2}^3.$$

Using the Schwarz and Young's inequalities, we have

$$(3.80) \quad \begin{aligned} \|\Phi^m\|_{L^2}^3 &= \left( \|\Phi^m\|_{L^2}^2 \right)^{\frac{3}{2}} = a_h(-\Delta_h^{-1} \Phi^m, \Phi^m)^{\frac{3}{2}} \\ &\leq a_h(\Delta_h^{-1} \Phi^m, \Delta_h^{-1} \Phi^m)^{\frac{3}{4}} a_h(\Phi^m, \Phi^m)^{\frac{3}{4}} \\ &\leq \epsilon^{\frac{4(1+d)}{4-d} + \sigma_1 + 2(d-2)\sigma_1} \frac{\epsilon^4}{1-\epsilon^3} a_h(\Phi^m, \Phi^m) \\ &\quad + C\epsilon^{-4} \epsilon^{-\frac{4(1+d)}{4-d} - \sigma_1 - 2(d-2)\sigma_1} a_h(\Delta_h^{-1} \Phi^m, \Delta_h^{-1} \Phi^m)^3. \end{aligned}$$

Therefore, (3.78) becomes

$$\begin{aligned}
(3.81) \quad & a_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell) + \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}, \Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}) \\
& + \frac{\epsilon^4 k}{1 - \epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 \\
& \leq Ck \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) \\
& + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m)^3 \\
& + C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)) + C\left(h^6\epsilon^{-6}\|u\|_{L^2((0,T);H^s(\Omega))}^2 + k^2\epsilon^{-6}E_h(U^0)\right) \\
& \leq Ck \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) \\
& + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m)^3 \\
& + C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)).
\end{aligned}$$

On noting that  $U^m$  can be written as

$$(3.82) \quad U^\ell = k \sum_{m=1}^{\ell} d_t U^m + U^0,$$

then by (2.3) and (3.17), we get

$$(3.83) \quad \|U^\ell\|_{-1,h} \leq k \sum_{m=1}^{\ell} \|d_t U^m\|_{-1,h} + \|U^0\|_{-1,h} \leq C\epsilon^{-\sigma_1}.$$

Using the boundedness of the projection, we have

$$(3.84) \quad \|\Phi^\ell\|_{-1,h}^2 \leq C\epsilon^{-2\sigma_1}.$$

Also, (3.81) can be written in the following equivalent form

$$\begin{aligned}
(3.85) \quad & a_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell) + \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}, \Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}) \\
& + \frac{\epsilon^4 k}{1 - \epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{2k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 \leq M_1 + M_2,
\end{aligned}$$

where

$$\begin{aligned}
(3.86) \quad M_1 &:= Ck \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) \\
&\quad + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m)^3 \\
&\quad + C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)),
\end{aligned}$$

$$\begin{aligned}
(3.87) \quad M_2 &:= Cka_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell) \\
&\quad + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} a_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell)^3.
\end{aligned}$$

It is easy to check that

$$(3.88) \quad M_2 \leq \frac{1}{2} \|\Phi^\ell\|_{-1,h}^2 \quad \text{provided that} \quad k \leq C\epsilon^{\frac{4(6+d)}{4-d}+(4d-2)\sigma_1}.$$

Under this restriction, we have

$$\begin{aligned}
(3.89) \quad &a_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell) + 2 \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}, \Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}) \\
&\quad + \frac{2\epsilon^4 k}{1-\epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{4k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 \\
&\leq 2Ck \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) + 2C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)) \\
&\quad + 2Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m)^3 \\
&\leq Ck \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) + C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)) \\
&\quad + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m)^3.
\end{aligned}$$

Define the slack variable  $d_\ell \geq 0$  such that

$$\begin{aligned}
(3.90) \quad &a_h(\Delta_h^{-1}\Phi^\ell, \Delta_h^{-1}\Phi^\ell) + 2 \sum_{m=1}^{\ell} a_h(\Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}, \Delta_h^{-1}\Phi^m - \Delta_h^{-1}\Phi^{m-1}) \\
&\quad + \frac{2\epsilon^4 k}{1-\epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{4k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4 + d_\ell \\
&= Ck \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m) + C(k^2\rho_1(\epsilon; d) + h^6\rho_3(\epsilon)) \\
&\quad + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} \sum_{m=1}^{\ell-1} a_h(\Delta_h^{-1}\Phi^m, \Delta_h^{-1}\Phi^m)^3.
\end{aligned}$$

We also define  $\{S_\ell\}_{\ell \geq 1}$  by

$$(3.91) \quad S_\ell = d_\ell + 2 \sum_{m=1}^{\ell} a_h(\Delta_h^{-1} \Phi^m - \Delta_h^{-1} \Phi^{m-1}, \Delta_h^{-1} \Phi^m - \Delta_h^{-1} \Phi^{m-1}) \\ + a_h(\Delta_h^{-1} \Phi^\ell, \Delta_h^{-1} \Phi^\ell) + \frac{2\epsilon^4 k}{1 - \epsilon^3} \sum_{m=1}^{\ell} a_h(\Phi^m, \Phi^m) + \frac{4k}{\epsilon} \sum_{m=1}^{\ell} \|\Phi^m\|_{L^4}^4,$$

and equation (3.90) shows that

$$S_1 = C(k^2 \rho_1(\epsilon; d) + h^6 \rho_3(\epsilon)).$$

Then

$$(3.92) \quad S_{\ell+1} - S_\ell \leq CkS_\ell + Ck\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} S_\ell^3 \quad \forall \ell \geq 1.$$

Applying Lemma 2.3 to  $\{S_\ell\}_{\ell \geq 1}$  defined above, we obtain  $\forall \ell \geq 1$ ,

$$(3.93) \quad S_\ell \leq a_\ell^{-1} \left\{ S_1^{-2} - 2C\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} k \sum_{s=1}^{\ell-1} a_{s+1}^{-2} \right\}^{-\frac{1}{2}}$$

provided that

$$(3.94) \quad S_1^{-2} - 2C\epsilon^{-\frac{4(6+d)}{4-d}-2\sigma_1-4(d-2)\sigma_1} k \sum_{s=1}^{\ell-1} a_{s+1}^{-2} > 0.$$

We note that  $a_s$  ( $1 \leq s \leq \ell$ ) are all bounded as  $k \rightarrow 0$ , therefore, (3.94) holds under the mesh constraint stated in the theorem. It follows from (3.51) and (3.52) that

$$(3.95) \quad S_\ell \leq 2a_\ell^{-1} S_1 \leq C(k^2 \rho_1(\epsilon; d) + h^6 \rho_3(\epsilon)).$$

Then (3.53) follows from the triangle inequality on  $E^m = \Theta^m + \Phi^m$ . (3.55) is obtained by taking the test function  $\eta_h = \Phi^m$  in (3.68) and  $v_h = \Phi^m$  in (3.69), and (3.54) is a consequence of the Poincaré inequality.

Now setting  $\eta_h = \Phi^m$  in (3.68) and  $v_h = -\frac{1}{\epsilon} \Psi^m$  in (3.69), and adding the resulting equations yield

$$(3.96) \quad \frac{1}{2} d_t \|\Phi^m\|_{L^2}^2 + \frac{k}{2} \|d_t \Phi^m\|_{L^2}^2 + \frac{1}{\epsilon} \|\Psi^m\|_{L^2}^2 = \frac{1}{\epsilon^2} (f(u(t_m)) - f(U^m), \Psi^m) \\ + (R(u_{tt}; m), \Phi^m) - (d_t \Theta^m, \Phi^m) - \frac{1}{\epsilon} (\Lambda^m, \Psi^m).$$

The last three terms on the right-hand side of (3.96) can be bounded in the same way as in (3.71), and the first term can be controlled as

$$(3.97) \quad \frac{1}{\epsilon^2} (f(u(t_m)) - f(U^m), \Psi^m) = \frac{1}{\epsilon^2} (f'(\xi) E^m, \Psi^m) \\ \leq \frac{1}{2\epsilon} \|\Psi^m\|_{L^2}^2 + \frac{C}{\epsilon^3} \|E^m\|_{L^2}^2.$$

Multiplying both sides of (3.96) by  $k$  and summing over  $m$  from 1 to  $M$  yield the desired estimate (3.57). Estimate (3.58) follows from an applications of the following inverse inequality:

$$(3.98) \quad \|\Phi^m\|_{L^\infty} \leq h^{-\frac{d}{2}} \|\Phi^m\|_{L^2},$$

and the following  $L^\infty$  estimate for the elliptic projection:

$$(3.99) \quad \|u - P_h u\|_{L^\infty} \leq Ch^2 |\ln h| \|u\|_{W^{s,\infty}} \quad \forall u \in H^2(\Omega).$$

Finally, it is well known that there holds the following estimate for the elliptic projection operator:

$$(3.100) \quad \max_{0 \leq m \leq M} \|\Lambda^m\|_{L^2} + \left( k \sum_{m=0}^M k \|d_t \Lambda^m\|_{L^2}^2 \right)^{\frac{1}{2}} \leq Ch^2 \rho_2(\epsilon).$$

Using the identity

$$(3.101) \quad (d_t \Phi^m, \Phi^m) = \frac{1}{2} d_t \|\Phi^m\|_{L^2}^2 + \frac{k}{2} \|d_t \Phi^m\|_{L^2}^2,$$

we get

$$(3.102) \quad \begin{aligned} \frac{1}{2} \|\Psi^M\|_{L^2}^2 + k \sum_{m=1}^M \frac{k}{2} \|d_t \Psi^m\|_{L^2}^2 &= k \sum_{m=1}^M (d_t \Psi^m, \Psi^m) + \frac{1}{2} \|\Psi^0\|_{L^2}^2 \\ &\leq k \sum_{m=1}^M \left( \frac{k}{4} \|d_t \Psi^m\|_{L^2}^2 + \frac{1}{k} \|\Psi^m\|_{L^2}^2 \right) + \frac{1}{2} \|\Psi^0\|_{L^2}^2. \end{aligned}$$

The first term on the right hand side of (3.102) can be absorbed by the second term on the left hand side of (3.102). The second term on the right hand side of (3.102) has been obtained in (3.57). Estimate (3.61) for  $W^m$  then follows from (3.100) and (3.102). (3.62) follows from an application of the triangle inequality, the inverse inequality, and (3.99). This completes the proof.  $\square$

**4. Convergence of numerical interfaces.** In this section, we prove that the numerical interface defined as the zero level set of the finite element interpolation of the solution  $U^m$  converges to the moving interface of the Hele-Shaw problem under the assumption that the Hele-Shaw problem has a unique global (in time) classical solution. To the end, we first cite the following PDE convergence result proved in [2].

**THEOREM 4.1.** *Let  $\Omega$  be a given smooth domain and  $\Gamma_{00}$  be a smooth closed hypersurface in  $\Omega$ . Suppose that the Hele-Shaw problem starting from  $\Gamma_{00}$  has a unique smooth solution  $(w, \Gamma := \bigcup_{0 \leq t \leq T} (\Gamma_t \times \{t\}))$  in the time interval  $[0, T]$  such that  $\Gamma_t \subseteq \Omega$  for all  $t \in [0, T]$ . Then there exists a family of smooth functions  $\{u_0^\epsilon\}_{0 < \epsilon \leq 1}$  which are uniformly bounded in  $\epsilon \in (0, 1]$  and  $(x, t) \in \bar{\Omega}_T$ , such that if  $u^\epsilon$  solves the Cahn-Hilliard problem (1.1)–(1.5), then*

- (i)  $\lim_{\epsilon \rightarrow 0} u^\epsilon(x, t) = \begin{cases} 1 & \text{if } (x, t) \in \mathcal{O} \\ -1 & \text{if } (x, t) \in \mathcal{I} \end{cases}$  uniformly on compact subsets, where  $\mathcal{I}$  and  $\mathcal{O}$  stand for the “inside” and “outside” of  $\Gamma$ ;
- (ii)  $\lim_{\epsilon \rightarrow 0} (\epsilon^{-1} f(u^\epsilon) - \epsilon \Delta u^\epsilon)(x, t) = -w(x, t)$  uniformly on  $\bar{\Omega}_T$ .

We note that since  $U^m$  is multi-valued on the edges of the mesh  $\mathcal{T}_h$ , its zero-level set is not well defined. To avoid this technicality, we use a continuous finite element interpolation of  $U^m$  to define the numerical interface. Let  $\hat{U}^m \in S_h$  denote the finite element approximation of  $U^m$  which is defined using the averaged degrees of freedom of  $U^m$  as the degrees of freedom for determining  $\hat{U}^m$  (cf. [17]). The following approximation results were proved in Theorem 2.1 of [17].

**THEOREM 4.2.** *Let  $\mathcal{T}_h$  be a conforming mesh consisting of triangles when  $d = 2$ , and tetrahedra when  $d = 3$ . For  $v_h \in V_h$ , let  $\hat{v}_h$  be the finite element approximation*

of  $v_h$  as defined above. Then for any  $v_h \in V_h$  and  $i = 0, 1$  there holds

$$(4.1) \quad \sum_{K \in \mathcal{T}_h} \|v_h - \widehat{v}_h\|_{H^i(K)}^2 \leq C \sum_{e \in \mathcal{E}_h^I} h_e^{1-2i} \|[v_h]\|_{L^2(e)}^2,$$

where  $C > 0$  is a constant independent of  $h$  and  $v_h$  but may depend on  $r$  and the minimal angle  $\theta_0$  of the triangles in  $\mathcal{T}_h$ .

By the construction,  $\widehat{U}^m$  is expected to be very close to  $U^m$ , hence,  $\widehat{U}^m$  should also be very close to  $u(t_m)$ . This is indeed the case as stated in the following theorem, which says that Theorem 3.6 also hold for  $\widehat{U}^m$ .

**THEOREM 4.3.** *Let  $U^m$  denote the solution of scheme (3.1)–(3.14) and  $\widehat{U}^m$  denote its finite element approximation as defined above. Then under the assumptions of Theorem 3.6 the error estimates for  $U^m$  given in Theorem 3.6 are still valid for  $\widehat{U}^m$ , in particular, there holds*

$$(4.2) \quad \max_{0 \leq m \leq M} \|u(t_m) - \widehat{U}^m\|_{L^\infty(\mathcal{T}_h)} \leq C \left( h^2 |\ln h| \epsilon^{-\gamma} + h^{-\frac{d}{2}} \epsilon^{-\frac{7}{2}} r(h, k; \epsilon, d, \sigma_i)^{\frac{1}{2}} \right).$$

We omit the proof to save space and refer the reader to [13] to see a proof of the same nature for the related Allen-Cahn problem.

We are now ready to state the first main theorem of this section.

**THEOREM 4.4.** *Let  $\{\Gamma_t\}_{t \geq 0}$  denote the zero level set of the Hele-Shaw problem and  $(U_{\epsilon, h, k}(x, t), W_{\epsilon, h, k}(x, t))$  denote the piecewise linear interpolation in time of the finite element interpolation  $\{(\widehat{U}^m, \widehat{W}^m)\}$  of the DG solution  $\{(U^m, W^m)\}$ , namely,*

$$(4.3) \quad U_{\epsilon, h, k}(x, t) := \frac{t - t_{m-1}}{k} \widehat{U}^m(x) + \frac{t_m - t}{k} \widehat{U}^{m-1}(x),$$

$$(4.4) \quad W_{\epsilon, h, k}(x, t) := \frac{t - t_{m-1}}{k} W^m(x) + \frac{t_m - t}{k} W^{m-1}(x),$$

for  $t_{m-1} \leq t \leq t_m$  and  $1 \leq m \leq M$ . Then, under the mesh and starting value constraints of Theorem 3.6 and  $k = O(h^{2-\gamma})$  with  $\gamma > 0$ , we have

- (i)  $U_{\epsilon, h, k}(x, t) \xrightarrow{\epsilon \searrow 0} 1$  uniformly on compact subset of  $\mathcal{O}$ ,
- (ii)  $U_{\epsilon, h, k}(x, t) \xrightarrow{\epsilon \searrow 0} -1$  uniformly on compact subset of  $\mathcal{I}$ .
- (iii) Moreover, in the case that dimension  $d = 2$ , when  $k = O(h^3)$ , suppose that  $W^0$  satisfies  $\|w_0^\epsilon - W^0\|_{L^2} \leq Ch^\beta$  for some  $\beta > \frac{3}{2}$ , then we have  $W_{\epsilon, h, k}(x, t) \xrightarrow{\epsilon \searrow 0} -w(x, t)$  uniformly on  $\overline{\Omega}_T$ .

*Proof.* For any compact set  $A \subset \mathcal{O}$  and for any  $(x, t) \in A$ , we have

$$(4.5) \quad \begin{aligned} |U_{\epsilon, h, k} - 1| &\leq |U_{\epsilon, h, k} - u^\epsilon(x, t)| + |u^\epsilon(x, t) - 1| \\ &\leq |U_{\epsilon, h, k} - u^\epsilon(x, t)|_{L^\infty(\Omega_T)} + |u^\epsilon(x, t) - 1|. \end{aligned}$$

Equation (3.58) of Theorem 3.6 infers that there exists a constant  $0 < \alpha < \frac{4-d}{2}$  such that

$$(4.6) \quad |U_{\epsilon, h, k} - u^\epsilon(x, t)|_{L^\infty(\Omega_T)} \leq Ch^\alpha.$$

The first term on the right-hand side of (4.5) tends to 0 when  $\epsilon \searrow 0$  (note that  $h, k \searrow 0$ , too). The second term converges uniformly to 0 on the compact set  $A$ , which is ensured by (i) of Theorem 4.1. Hence, the assertion (i) holds.



To show (ii), we only need to replace  $\mathcal{O}$  by  $\mathcal{I}$  and 1 by  $-1$  in the above proof. To prove (iii), under the assumptions  $k = O(h^3)$ , (3.62) in Theorem 3.6 implies that there exists a positive constant  $0 < \zeta < \frac{4-d}{2}$  such that

$$(4.7) \quad \|W_{\epsilon,h,k} - w^\epsilon\|_{L^\infty(\Omega_T)} \leq Ch^\zeta.$$

Then by the triangle inequality we obtain for any  $(x, t) \in \overline{\Omega}_T$ ,

$$(4.8) \quad |W_{\epsilon,h,k}(x, t) - (-w)| \leq |W_{\epsilon,h,k}(x, t) - w^\epsilon(x, t)| + |w^\epsilon(x, t) - (-w)|, \\ \leq \|W_{\epsilon,h,k}(x, t) - w^\epsilon(x, t)\|_{L^\infty(\Omega_T)} + |w^\epsilon(x, t) - (-w)|.$$

The first term on the right-hand side of (4.8) tends to 0 when  $\epsilon \searrow 0$  (note that  $h, k \searrow 0$ , too). The second term converges uniformly to 0 in  $\overline{\Omega}_T$ , which is ensured by (ii) of Theorem 4.1. Thus the assertion (iii) is proved. The proof is complete.  $\square$

The second main theorem of this section which is given below addresses the convergence of numerical interfaces.

**THEOREM 4.5.** *Let  $\Gamma_t^{\epsilon,h,k} := \{x \in \Omega; U_{\epsilon,h,k}(x, t) = 0\}$  be the zero level set of  $U_{\epsilon,h,k}(x, t)$ , then under the assumptions of Theorem 4.4, we have*

$$\sup_{x \in \Gamma_t^{\epsilon,h,k}} \text{dist}(x, \Gamma_t) \xrightarrow{\epsilon \searrow 0} 0 \quad \text{uniformly on } [0, T].$$

*Proof.* For any  $\eta \in (0, 1)$ , define the open tabular neighborhood  $\mathcal{N}_\eta$  of width  $2\eta$  of  $\Gamma_t$  as

$$(4.9) \quad \mathcal{N}_\eta := \{(x, t) \in \Omega_T; \text{dist}(x, \Gamma_t) < \eta\}.$$

Let  $A$  and  $B$  denote the complements of the neighborhood  $\mathcal{N}_\eta$  in  $\mathcal{O}$  and  $\mathcal{I}$ , respectively, i.e.

$$A = \mathcal{O} \setminus \mathcal{N}_\eta \quad \text{and} \quad B = \mathcal{I} \setminus \mathcal{N}_\eta.$$

Note that  $A$  is a compact subset outside  $\Gamma_t$  and  $B$  is a compact subset inside  $\Gamma_t$ , then there exists  $\epsilon_3 > 0$ , which only depends on  $\eta$ , such that for any  $\epsilon \in (0, \epsilon_3)$

$$(4.10) \quad |U_{\epsilon,h,k}(x, t) - 1| \leq \eta \quad \forall (x, t) \in A,$$

$$(4.11) \quad |U_{\epsilon,h,k}(x, t) + 1| \leq \eta \quad \forall (x, t) \in B.$$

Now for any  $t \in [0, T]$  and  $x \in \Gamma_t^{\epsilon,h,k}$ , from  $U_{\epsilon,h,k}(x, t) = 0$  we have

$$(4.12) \quad |U_{\epsilon,h,k}(x, t) - 1| = 1 \quad \forall (x, t) \in A,$$

$$(4.13) \quad |U_{\epsilon,h,k}(x, t) + 1| = 1 \quad \forall (x, t) \in B.$$

(4.10) and (4.12) imply that  $(x, t)$  is not in  $A$ , and (4.11) and (4.13) imply that  $(x, t)$  is not in  $B$ , then  $(x, t)$  must lie in the tubular neighborhood  $\mathcal{N}_\eta$ . Therefore, for any  $\epsilon \in (0, \epsilon_3)$ ,

$$(4.14) \quad \sup_{x \in \Gamma_t^{\epsilon,h,k}} \text{dist}(x, \Gamma_t) \leq \eta \quad \text{uniformly on } [0, T].$$

The proof is complete.  $\square$

**5. Numerical experiments.** In this section, we present three two-dimensional numerical tests to gauge the performance of the proposed fully discrete MIP-DG methods using the linear element (i.e.,  $r = 1$ ). The square domain  $\Omega = [-1, 1]^2$  is used in all three tests and the initial condition is chosen to have the form  $u_0 = \tanh(\frac{d_0(x)}{\sqrt{2\epsilon}})$ , where  $d_0(x)$  denotes the signed distance from  $x$  to the initial interface  $\Gamma_0$ .

Our first test uses a smooth initial condition to satisfy the requirement for  $u_0$ , consequently, the theoretical results established in this paper apply to this test problem. On the other hand, non-smooth initial conditions are used in the second and third tests, hence, the theoretical results of this paper may not apply. But we still use our MIP-DG methods to compute the error order, energy decay and the evolution of the numerical interfaces. Our numerical results suggest that the proposed DG schemes work well, even a convergence theory is missing for them.

**Test 1.** Consider the Cahn-Hilliard problem (1.1)-(1.5) with the following initial condition:

$$u_0(x) = \tanh\left(\frac{d_0(x)}{\sqrt{2\epsilon}}\right),$$

where  $\tanh(t) = (e^t - e^{-t})/(e^t + e^{-t})$ , and  $d_0(x)$  represents the signed distance function to the ellipse:

$$\frac{x_1^2}{0.36} + \frac{x_2^2}{0.04} = 1.$$

Hence,  $u_0$  has the desired form as stated in Proposition 3.5.

Table 5.1 shows the spatial  $L^2$  and  $H^1$ -norm errors and convergence rates, which are consistent with what are proved for the linear element in the convergence theorem.  $\epsilon = 0.1$  is used to generate the table.

	$L^\infty(L^2)$ error	$L^\infty(L^2)$ order	$L^2(H^1)$ error	$L^2(H^1)$ order
$h = 0.4\sqrt{2}$	0.53325		0.84260	
$h = 0.2\sqrt{2}$	0.21280	1.3253	0.64843	0.3779
$h = 0.1\sqrt{2}$	0.07164	1.5707	0.43273	0.5835
$h = 0.05\sqrt{2}$	0.01779	2.0097	0.21411	1.0151
$h = 0.025\sqrt{2}$	0.00454	1.9703	0.10890	0.9753

TABLE 5.1  
Spatial errors and convergence rates of Test 1 with  $\epsilon = 0.1$ .

Figure 5.1 plots the change of the discrete energy  $E_h(U^\ell)$  in time, which should decrease according to (3.17). This graph clearly confirms this decay property. Figure 5.2 displays four snapshots at four fixed time points of the numerical interface with four different  $\epsilon$ . They clearly indicate that at each time point the numerical interface converges to the sharp interface  $\Gamma_t$  of the Hele-Shaw flow as  $\epsilon$  tends to zero. It also shows that the numerical interface evolves faster in time for larger  $\epsilon$  and confirms the mass conservation property of the Cahn-Hilliard problem as the total mass does not change in time, which approximates a constant 3.064.

**Test 2.** Consider the Cahn-Hilliard problem (1.1)-(1.5) with the following initial condition:

$$u_0(x) = \tanh\left(\frac{1}{\sqrt{2\epsilon}}\left(\min\left\{\sqrt{(x_1 + 0.3)^2 + x_2^2} - 0.3, \sqrt{(x_1 - 0.3)^2 + x_2^2} - 0.25\right\}\right)\right).$$

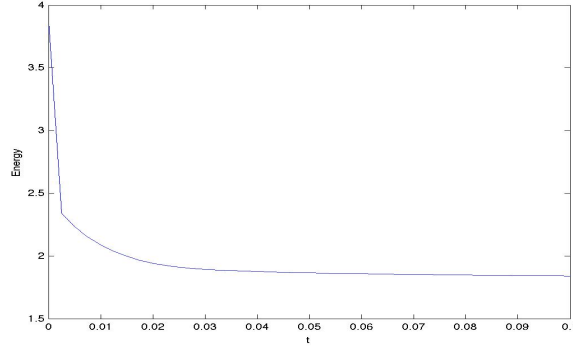


FIG. 5.1. *Decay of the numerical energy  $E_h(U^\ell)$  of Test 1.*

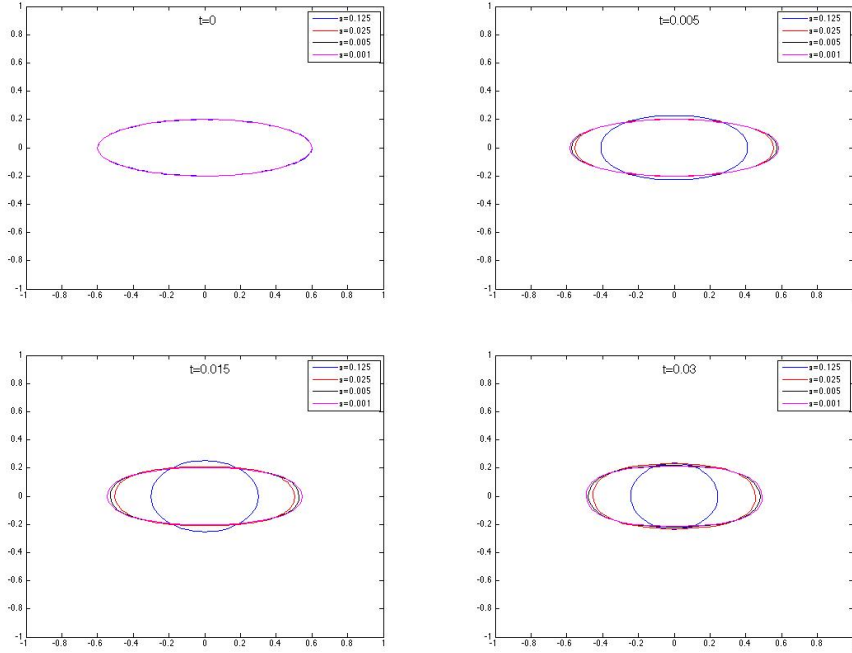


FIG. 5.2. *Test 1: Snapshots of the zero-level set of  $u^{\epsilon,h,k}$  at time  $t = 0, 0.005, 0.015, 0.03$  and  $\epsilon = 0.125, 0.025, 0.005, 0.001$ .*

We note that  $u_0$  can be written as

$$u_0(x) = \tanh\left(\frac{d_0(x)}{\sqrt{2}\epsilon}\right).$$

Here  $d_0(x)$  represents the signed distance function. We note that  $u_0$  does not have the desired form as stated in Proposition 3.5.

Table 5.2 shows the spatial  $L^2$  and  $H^1$ -norm errors and convergence rates, which are consistent with what are proved for the linear element in the convergence theorem.  $\epsilon = 0.1$  is used to generate the table. Figure 5.3 plots the change of the discrete energy

	$L^\infty(L^2)$ error	$L^\infty(L^2)$ order	$L^2(H^1)$ error	$L^2(H^1)$ order
$h = 0.4\sqrt{2}$	0.26713		0.35714	
$h = 0.2\sqrt{2}$	0.07161	1.8993	0.18411	0.9559
$h = 0.1\sqrt{2}$	0.01833	1.9660	0.09620	0.9365
$h = 0.05\sqrt{2}$	0.00476	1.9452	0.04928	0.9650
$h = 0.025\sqrt{2}$	0.00121	1.9760	0.02497	0.9808

TABLE 5.2  
Spatial errors and convergence rates of Test 2 with  $\epsilon = 0.1$ .

$E_h(U^\ell)$  in time, which should decrease according to (3.17). This graph clearly confirms this decay property. Figure 5.4 displays four snapshots at four fixed time points of

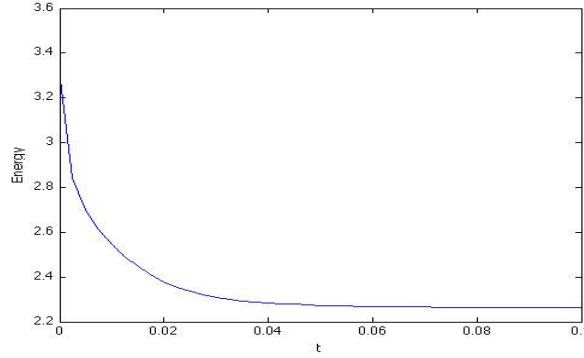


FIG. 5.3. Decay of the numerical energy  $E_h(U^\ell)$  of Test 2.

the numerical interface with four different  $\epsilon$ . They clearly indicate that at each time point the numerical interface converges to the sharp interface  $\Gamma_t$  of the Hele-Shaw flow as  $\epsilon$  tends to zero. It again shows that the numerical interface evolves faster in time for larger  $\epsilon$  and confirms the mass conservation property of the Cahn-Hilliard problem as the total mass does not change in time, which approximates a constant 3.032.

**Test 3.** Consider the Cahn-Hilliard problem (1.1)–(1.5) with the following initial condition:

$$u_0(x) = \tanh\left(\frac{1}{\sqrt{2}\epsilon} \left( \min\left\{ \sqrt{(x_1 + 0.3)^2 + x_2^2} - 0.2, \sqrt{(x_1 - 0.3)^2 + x_2^2} - 0.2, \sqrt{x_1^2 + (x_2 + 0.3)^2} - 0.2, \sqrt{x_1^2 + (x_2 - 0.3)^2} - 0.2 \right\} \right)\right).$$

Notice that the above  $u_0$  does not have the desired form as stated in Proposition 3.5.

Table 5.3 shows the spatial  $L^2$  and  $H^1$ -norm errors and convergence rates with  $\epsilon = 0.1$ , which are consistent with what are proved for the linear element in the convergence theorem. Figure 5.5 plots the change of the discrete energy  $E_h(U^\ell)$  in time, which again decreases as predicted by (3.17). Figure 5.6 displays four snapshots at four fixed time points of the numerical interface with four different  $\epsilon$ . Once again, we observe that at each time point the numerical interface converges to the sharp interface  $\Gamma_t$  of the Hele-Shaw flow as  $\epsilon$  tends to zero, the interface evolves faster in

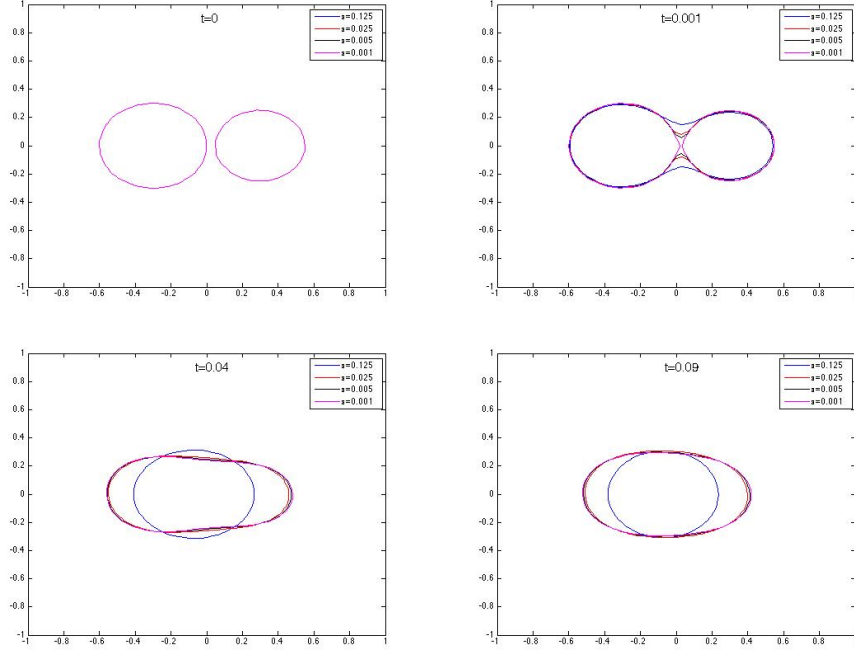


FIG. 5.4. *Test 2: Snapshots of the zero-level set of  $u^{\epsilon,h,k}$  at time  $t = 0, 0.001, 0.04, 0.09$  and  $\epsilon = 0.125, 0.025, 0.005, 0.001$ .*

	$L^\infty(L^2)$ error	$L^\infty(L^2)$ order	$L^2(H^1)$ error	$L^2(H^1)$ order
$h = 0.4\sqrt{2}$	0.38576		0.84157	
$h = 0.2\sqrt{2}$	0.12347	1.6435	0.55082	0.6115
$h = 0.1\sqrt{2}$	0.03599	1.7785	0.31149	0.8224
$h = 0.05\sqrt{2}$	0.00965	1.8990	0.16199	0.9433
$h = 0.025\sqrt{2}$	0.00247	1.9660	0.08218	0.9790

TABLE 5.3  
*Spatial errors and convergence rates of Test 3 with  $\epsilon = 0.1$ .*

time for larger  $\epsilon$  and the mass conservation property is preserved. The total mass approximates a constant 2.989.

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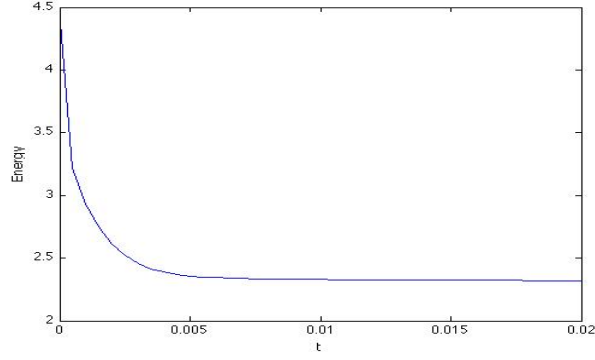


FIG. 5.5. Decay of the numerical energy  $E_h(U^\ell)$  of Test 3.

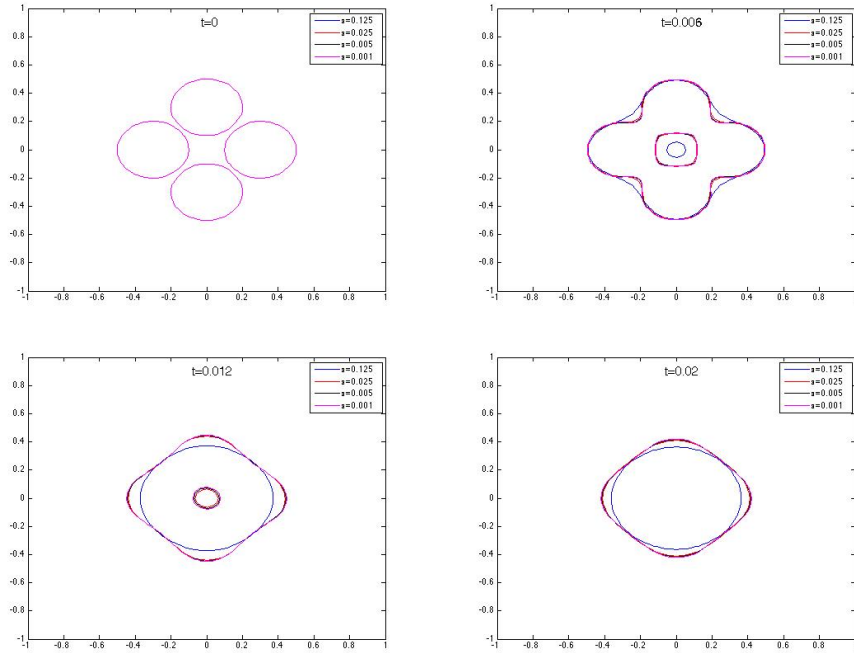


FIG. 5.6. Test 3: Snapshots of the zero-level set of  $u^{\epsilon,h,k}$  at time  $t = 0, 0.006, 0.012, 0.02$  and  $\epsilon = 0.125, 0.025, 0.005, 0.001$ .

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